7.3 Subbotovskaya's method

- Bella Abramovna Subbotovskaya, Realizations of linear functions by formulas using $+, \cdot, \overline{}$, Sov. Math. Dokl. 2 (1961), 110 - 112.

Idea:

Given a standard formula β computing some function $f \in B_n$ set randomly some of the variables to constants and show that this restriction reduces the size of β considerably whereas the resulting subfunction of f is not much easier.

Lemma 7.2

Let $f \in B_n$. It is possible to fix one of its variables such that the resulting function $f' \in B_{n-1}$ satisfies

$$L_{\Omega_0}(f') \leq \left(1 - \frac{1}{n}\right)^{3/2} L_{\Omega_0}(f).$$

Proof:

Let β be an optimal standard formula computing f and let

$$s := L_{\Omega_0}(f).$$

Pidgeon hole principle $\Rightarrow$

$\exists$ variable x_i which appears $\geq \frac{s}{n}$ times as a leaf; i.e.,

$\geq \frac{s}{n}$ leaves are labeled by x_i or $\neg x_i$.

$\Rightarrow$

After setting x_i to a constant $c \in \{0,1\}$, we obtain a standard formula β' for a function in B_{n-1} with at most

$$s - \frac{s}{n} = \left(1 - \frac{1}{n}\right) s$$

leaves.

Moreover, we shall see that some other leaves are eliminated by this setting as well.

If for a leaf z, $z \wedge \tilde{\beta}$ or $z \vee \tilde{\beta}$ is a sub=formula of β then we say that the subformula $\tilde{\beta}$ is a <u>neighbour</u> of the leaf z.

<u>Claim</u> (+):

If $z \in \{x_i, \neg x_i\}$ is a leaf of β then the neighbour of z does not contain the variable x_i.

<u>Proof of claim</u>.

Note that β is an optimal standard formula. Let $z \in \{x_i, \neg x_i\}$ be a leaf of β with neighbour $\tilde{\beta}$.

$$\Rightarrow \quad \hat{\beta} = z \wedge \tilde{\beta} \quad \text{or} \quad \hat{\beta} = z \vee \tilde{\beta}$$

is a subformula of β.

Assume that $\tilde{\beta}$ contains a leaf $z' \in \{x_i, \neg x_i\}$.

Replace this leaf by that constant $c \in \{0,1\}$ for which the literal

$$\begin{cases} z \text{ gets the value } 1 & \text{if } \hat{\beta} = z \wedge \tilde{\beta} \\ z \text{ gets the value } 0 & \text{if } \hat{\beta} = z \vee \tilde{\beta} \end{cases}$$

$\leadsto$

$\hat{\beta} = z \wedge \tilde{\beta}$:

$$z' := \begin{cases} 1 & \text{if } z' = z \\ 0 & \text{if } z' \neq z \end{cases}$$

$\hat{\beta} = z \vee \tilde{\beta}$

$$z' := \begin{cases} 1 & \text{if } z' \neq z \\ 0 & \text{if } z' = z \end{cases}$$

After this setting, the following hold:

i) The resulting subformula $\tilde{\beta}'$ has one leaf fewer than $\tilde{\beta}$.

ii) The resulting subformula $\hat{\beta}'$ computes the same Boolean function as $\hat{\beta}$.

<u>Exercise</u>

Show that $\hat{\beta}'$ and $\hat{\beta}$ compute the same Boolean function.

$\Rightarrow$ The obtained formula β' computes the same Boolean function as β but has one leaf fewer.

This contradicts the optimality of β.

$\square$

Consider now a variable x_i which appears

$$t \geq \frac{s}{n}$$

times as a leaf of β. Let $z_1, z_2, \dots, z_t$ be the leaves labeled by x_i or by $\neg x_i$.

Claim $\Rightarrow$

For each $i = 1, 2, \dots, t$ there is a constant $c_i \in \{0,1\}$ such that after setting $x_i := c_i$ the neighbour of z_i would disappear from β. such that at least one further leaf which is not among $z_1, z_2, \dots, z_t$ would be erased.

Let $c \in \{0,1\}$ be that constant which appears most often in the sequence $c_1, c_2, \dots, c_t$.

If we set $x_i := c$ then all leaves $z_1, z_2, \dots, z_t$ would disappear from the formula and

$$\geq \frac{t}{2}$$

additional leaves would disappear as well.

$\Rightarrow$

$$\geq t + \frac{t}{2} \geq \frac{3}{2}\frac{s}{n}$$

leaves are eliminated after fixing $x_i := c$.

$\Rightarrow$

The resulting formula has at most

$$s - \frac{3}{2}\frac{s}{n} = s\left(1 - \frac{3}{2n}\right) \leq s\left(1-\frac{1}{n}\right)^{\frac{3}{2}}$$

leaves.

Hence, $L_{\Omega_0}(f') \leq \left(1-\frac{1}{n}\right)^{3/2} L_{\Omega_0}(f)$.

$\blacksquare$

<u>Theorem 7.8</u>

For every Boolean function $f \in B_n$ and all integer $1 \leq k \leq n$, it is possible to fix $n-k$ variables such that the resulting Boolean function $f' \in B_k$ satisfies

$$L_{\Omega_0}(f') \leq \left(\frac{k}{n}\right)^{3/2} L_{\Omega_0}(f).$$

<u>Proof:</u>

Let $s := L_{\Omega_0}(f)$. By applying Lemma 7.2 $n-k$ times, we obtain a formula depending on k variables with at most

$$\left(1-\frac{1}{k+1}\right)^{3/2}\cdot\left(1-\frac{1}{k+2}\right)^{3/2}\cdot\ldots\cdot\left(1-\frac{1}{n-1}\right)^{3/2}\left(1-\frac{1}{n}\right)^{3/2}\cdot s$$

$$\leq \left(\frac{k}{n}\right)^{3/2}\cdot s = \left(\frac{k}{n}\right)^{3/2} L_{\Omega_0}(f)$$

leaves.

$\blacksquare$

In order to prove larger lower bounds it is useful to restate Subbotovskaya's argument in probabilistic terms.

Let $f \in B_n$. A <u>partial assignment</u> (or <u>restriction</u>) ρ is a function

$$\rho : X_n \to \{0, 1, *\}$$

where $*$ means that the corresponding variable is unassigned.

Each partial assignment ρ yields a subfunction f_ρ of f in a natural way:

$$f_\rho = f\big(\rho(x_1), \rho(x_2), \ldots, \rho(x_n)\big).$$

Then f_ρ is a function of the variables x_i for which $\rho(x_i) = *$.

<u>Example</u>:

Consider

$$f = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2) \wedge (x_1 \vee \neg x_3)$$

and

$\rho(x_1) = 1, \quad \rho(x_2) = \rho(x_3) = *$

Then

$$f_\rho = x_2 .$$

Let R_k be the set of all partial assignments which leave exactly k variables unassigned.

What we shall be interested in is the random subfunctions that results from choosing a random partial assignment from R_k.

The probability distribution of restrictions in R_k which we shall use is the following:

(1) Randomly assign k variables to be $*$.
(2) Randomly and independently assign all other variables to be 0 or 1.

Theorem 7.8 $\Rightarrow$

$\forall f \in B_n$ there exists an assignment $\rho \in R_k$ such that

$$L_{\Omega_0}(f_\rho) \le \left(\frac{k}{n}\right)^{3/2} L_{\Omega_0}(f).$$

Goal:
To prove that a fraction of at least 3/4 of all assignments in R_k has this property.

Lemma 7.3

Let $f \in B_n$ and let ρ be a random assignment from R_k. Then, with probability at least $3/4$

$$L(f_\rho) \leq 4\left(\frac{k}{n}\right)^{3/2} L_{\Omega_0}(f).$$

Proof:

Let β be an optimal standard formula for f of size $s := L_{\Omega_0}(f)$.

Construct the assignment ρ in $n-k$ stages as follows:

- At any stage, choose a variable randomly from the remaining ones, and assign it 0 or 1 randomly.

We shall analyze the effect of this assignment to the formula β stage-by-stage.

Suppose that the first stage choose the variable x_i

$\leadsto$

After setting this variable to a constant, all leaves labeled by the literals x_i and $\neg x_i$ disappear from the formula β.

By averaging, the expected number of such literals is $\frac{s}{n}$.

Since x_i is assigned 0 or 1 randomly with equal probability $\frac{1}{2}$, we can expect by Claim (+)

$$\geq \frac{s}{2n} \quad \text{additional leaves disappear from } \beta.$$

$\Longrightarrow$

In total, the expected number of leaves which disappear in the first stage is

$$\geq \frac{s}{n} + \frac{s}{2n} = \frac{3s}{2n}$$

yielding a new formula with expected size

$$\leq s - \frac{3s}{2n} \leq \left(1 - \frac{1}{n}\right)^{3/2} \cdot s.$$

Subsequent stages of the assignment can be analyzed in the same way.

After each stage the number of variables decreases by one.

$\Longrightarrow$

After $n-k$ stages, the expected number of leaves of the final formula is at most

$$\left(1 - \frac{1}{k+1}\right)^{3/2}\left(1 - \frac{1}{k+2}\right)^{3/2} \cdots \left(1 - \frac{1}{n-1}\right)^{3/2}\left(1 - \frac{1}{n}\right)^{3/2} \cdot s$$

$$\leq \left(\frac{k}{n}\right)^{3/2} \cdot s.$$

By Markov's inequality, the probability that the random variable $L(\rho_s)$ is more than four times its expected value is smaller than $1/4$.

This completes the proof.

$\blacksquare$

Subbotovskaya's result can be stated for more general distributions.

Suppose that $0 < p < 1$. A <u>p-random assignment</u> independently assigns each variable x_i the value 0 or 1 with equal probabilities

$$\frac{1}{2}(1-p).$$

With the remaining probability p the variable x_i remains to be unassigned.

$\Rightarrow$

The distribution which we have considered in Lemma 7.3 corresponds to

$$p = \frac{k}{n}.$$

What is the expected formula size of the resulting function if we apply a p-random assignment?

The obvious answer to this question is

$$p \cdot L_{2,0}(f).$$

Subbotovskaya has shown that formulas shrink more. Namely, she has established an upper bound

$$O(p^{3/2} \cdot L_{\Omega_0}(f)).$$

on the expected formula size of the resulting function f_ρ.

Andreev's subsequent result has motivated the consideration of the shrinkage exponent Γ of standard formulas, which is defined to be the largest number such that for $f \in B_n$, the expected formula size of the resulting function after a p-random assignment is

$$O(p^{\Gamma} L_{\Omega_0}(f)).$$

Subbotovskaya showed $\Gamma \geq 1.5$.

- R. Impagliazzo, N. Nisan, The effect of random restrictions on formula size, Random Structures Algorithms 4 (1993), 121–134.

  show $\Gamma \geq 1.55$

- M. Paterson, U. Zwick, Shrinkage of deMorgan formulae under restriction, Random Structures Algorithms 4 (1993), 135–150.

  improve this to $\Gamma \geq 1.63$

- <u>Johan Håstad</u>, The shrinkage exponent of de Morgan formulas is 2, SIAM J. Comput. 27 (1998), 48–64.

  finally proved $\Gamma = 2$.

7.4 Andreev's method

<u>Alexander E. Andreev</u>, On a method for obtaining more than quadratic effective lower bounds for the complexity of π-schemes, Moscow Univ. Math. Bull. 42 (1987), 63–66.

Andreev's idea was to combine Subbotovskaya's with Nechiporuk's method of universal functions.

First we shall describe <u>universal functions</u> as introduced by Nechiporuk to prove quadratic lower bounds for the formula size of Boolean functions.

Let $n = 2^r$, $r \in \mathbb{N}$ and consider two sequences $z = (z_1, z_2, \dots, z_r)$ and $y = (y_1, y_2, \dots, y_n)$ of variables.

Each assignment $a \in \{0,1\}^r$ to the z-variables gives us the unique natural number $|a|$.

Consider $U_n \in B_{r+n}$ defined by

$$U_n(z, y) := y_{|z|}$$