The computation of the initial feasible basic solution is called Phase 1 of the simplex method.

The subsequent computation of an optimal solution is called Phase 2.

During the development of Phase 1 we have seen that in the case of a redundant system

$$Ax = b$$

of equations, one column and one row can be deleted without changing the polyhedron of feasible solutions.

We know what to do in the case that $\text{rank}(A) < m$.

**Exercise:**

Develop an algorithm for the solution of linear programs

$$\begin{align*}
\min & \quad z(x) = c^T x \\
Ax &= b \\
x &\geq 0
\end{align*}$$

with $A$ is an $(m \times n)$-matrix and $\text{rank}(A) < m$.

It remains the discussion what to do in the case of degenerate basic feasible solutions.
Assumption:

\[ \exists q \in \mathbb{R}, p \in \{1, 2, \ldots, m\} \text{ with } y_{po} = 0, y_{pq} > 0 \text{ and } y_{pq} > 0. \]

If the nonbasic variable \( x_p \) enters the basis then \( x_P \) or another basic variable with value 0 has to leave the basis. Although we get a new basis, the value of the objective function does not change. The new basic solution is also degenerate (note that \( y_{po} = 0 \) after the basis exchange).

Maybe, in the case of degenerate basic solutions, the algorithm SIMPLEX could be in an endless loop. This would be the case if the algorithm repeats a cycle of basic feasible solutions corresponding to the same vertex of the polyhedron infinitely often. This is called cycling.

If we have no restriction on the choice of the column which leaves the basis, there are examples where cycling occur.

\[ \frac{c}{j} \text{ goal:} \]

The development of selection rules for the column of the basis such that no cycling is possible.
For the development of such selection rules we shall use tables of Type 1. First we need some notations.

A vector \( \mathbf{v} \neq \mathbf{0} \) is lexicographical positive if its first \( + \mathbf{0} \)-component is positive. Then we write \( \mathbf{v} >_e \mathbf{0} \). A vector \( \mathbf{v} \) is called lexicographical larger than a vector \( \mathbf{w} \) if \( \mathbf{v} - \mathbf{w} >_e \mathbf{0} \).

A sequence \( \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^s \) of vectors with \( \mathbf{v}^{i+1} - \mathbf{v}^i >_e \mathbf{0} \) for \( 1 \leq i < s \) is called lexicographically increasing. \( \mathbf{v} >_e \mathbf{0} \) means \( \mathbf{v} = \mathbf{0} \) or \( \mathbf{v} >_e \mathbf{0} \). Analogously, we define lexicographically smaller and lexicographically decreasing.

To get the selection rule, we extend the table of Type 1 corresponding to the initial feasible basic solution in the following way:

**Extended table of Type 1.**

<table>
<thead>
<tr>
<th>( x_B_0 )</th>
<th>( y_0 )</th>
<th>0</th>
<th>\cdots</th>
<th>0</th>
<th>\cdots</th>
<th>0</th>
<th>\cdots</th>
<th>( y_{0j} )</th>
<th>\cdots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_B_1 )</td>
<td>( y_{10} )</td>
<td>1</td>
<td>\cdots</td>
<td>0</td>
<td>\cdots</td>
<td>0</td>
<td>\cdots</td>
<td>( y_{1j} )</td>
<td>\cdots</td>
</tr>
<tr>
<td></td>
<td>\ldots</td>
<td></td>
<td>\vdots</td>
<td></td>
<td>\vdots</td>
<td></td>
<td>\vdots</td>
<td>\ldots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( x_B_i )</td>
<td>( y_{i0} )</td>
<td>0</td>
<td>\cdots</td>
<td>1</td>
<td>\cdots</td>
<td>0</td>
<td>\cdots</td>
<td>( y_{ij} )</td>
<td>\cdots</td>
</tr>
<tr>
<td></td>
<td>\ldots</td>
<td></td>
<td>\vdots</td>
<td></td>
<td>\vdots</td>
<td></td>
<td>\vdots</td>
<td>\ldots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( x_B_m )</td>
<td>( y_{m0} )</td>
<td>0</td>
<td>\cdots</td>
<td>0</td>
<td>\cdots</td>
<td>1</td>
<td>\cdots</td>
<td>( y_{mj} )</td>
<td>\cdots</td>
</tr>
</tbody>
</table>

Between the solution column and the columns corresponding to the nonbasic variables, we add the \((m \times m)\)-unit matrix using the rows 1, 2, ..., \( m \). Row 0 obtains in the corresponding columns the value 0.
We shall use the additional columns only for the selection of the variables which leave the basis.

Let

\[ v_0 := (y_{00}, 0, 0, \ldots, 0) \quad \text{and} \]
\[ v_i := (y_{i0}, 0, \ldots, 0, 1, 0, \ldots, 0), \quad 1 \leq i \leq m \]

be the coefficients of the first \( m+1 \) columns of the rows \( 0, 1, 2, \ldots, m \) of the extended table.

Note that for \( 1 \leq i \leq m \)

\[ y_{i0} \geq 0 \quad \Rightarrow \quad v_i \geq 0. \]

Furthermore, these vectors are linear independent.

The selection rule will take care that the algorithm \textsc{Simplex} maintains the following invariant:

1. The vectors \( v_1, v_2, \ldots, v_m \) are lexicographical positive and linear independent.

Now we shall investigate the performance of one iteration of the algorithm \textsc{Simplex}.

**Assumption:**

\( y_{0q} > 0 \) and \( x_q \) is chosen to enter the basis.

Let

\[ S_q := \{ i \mid i > 1, y_{i0} > 0 \} \quad \text{and} \]
\[ u_i = \frac{v_i}{y_{i0}} \quad \text{for all } i \in S_q. \]
Let $u_p$ be the lexicographical smallest such a vector; i.e., $u_i \geq u_p$ for all $i \in S_q$.

We write then

$$u_p = \text{lexmin}\{u_i : i \in S_q\}.$$  

\[ \forall i, 1 \leq i \leq m \text{ linear independent } \Rightarrow u_p \text{ is uniquely determined} \]

- Choose the basic variable $x_{3_p}$ to leave the basis.

Since

$$u_p = \text{lexmin}\{u_i : i \in S_q\} \quad \Rightarrow \quad \frac{y_{p0}}{y_{p1}} = \min\left\{ \frac{y_{i0}}{y_{i1}} \mid i \in S_q \right\}$$

there holds:

$$x_{3_p} \text{ fulfills the necessary condition to leave the basis}.$$  

Let $\hat{v}_i, 1 \leq i \leq m$ be the vectors after the performance of the transformation.

To prove the maintenance of the invariant, we shall prove that

i) $\hat{v}_i, 1 \leq i \leq m$ are lexicographical positive

ii) $\hat{v}_i, 1 \leq i \leq m$ are linear independent.
Note that
\[ \hat{v}_p = \frac{v_p}{y_{pq}} \quad \text{and} \quad \hat{v}_i = v_i - \frac{y_{iq}}{y_{pq}} v_p = v_i - y_{iq} \hat{v}_p \quad \text{for } i \neq p. \]

Since \( v_p > \varepsilon > 0 \) and \( y_{pq} > 0 \) there hold \( \hat{v}_p > \varepsilon > 0 \).

- If \( y_{iq} < 0 \) then \( -y_{iq} \hat{v}_p > \varepsilon > 0 \) and therefore \( \hat{v}_i > \varepsilon > 0 \).
- If \( y_{iq} > 0 \) then because of the choice of \( u_p \)

\[ (*) \quad u_i - u_p = \frac{v_i}{y_{iq}} - \hat{v}_p > \varepsilon > 0. \]

After multiplication of equation \((*)\) with \( y_{iq} \) we obtain \( \hat{v}_i > \varepsilon > 0. \)

\[ \Rightarrow \]
\[ \hat{v}_i, \ 1 \leq i \leq m \text{ are lexicographical positive.} \]

Note that
- the addition of a multiple of a vector and another vector maintains the linear independence of a set of vectors

\[ \Rightarrow \]
\[ \hat{v}_i, \ 1 \leq i \leq m \text{ are linear independent.} \]

It remains to show that no cycling occurs.

Since the values in the table are completely determined by the corresponding basis,
until permutation of the nonbasic columns and permutation of the basic rows, the table of the $q$-th iteration depends only on the current basis $B_q$.

Hence, for $q < t$ there holds

$$v_q^t = v_0^t \implies B_q \neq B_t.$$  

If the sequence $\{v_0^t\}$ is lexicographically decreasing then

$$t_1 = t_2 \implies v_{t_1}^0 = v_{t_2}^0$$

such that no cycling occurs.

Note that

$$v_{t+1}^0 = v_0^t - \frac{y_{0q}^t}{y_{pq}^t} v_p^t$$

$$= v_0^t - y_{0q}^t v_p^{t+1}$$

Since $v_p^{t+1} > 0$ and $y_{0q}^t > 0$ there holds

$$v_{t+1}^0 < v_0^t.$$

$\implies$

The sequence $\{v_0^t\}$ is lexicographically decreasing.
3.3 Duality

Duality is a very important and helpful concept in combinatorial optimization. We have used this concept for the development of an efficient algorithm for the maximum weighted matching problem in bipartite graphs. With respect to linear programs, we obtain with help of duality for each linear program a corresponding other linear program which have some interesting properties. Let

\[ \text{LP: } \min z(x) = c^T x \]
\[ Ax = b \]
\[ x \geq 0 \]

be a linear program with \( \text{rank } (A) = m \) where \( m \) is the number of rows of \( A \). Let \( B \) be any basis of \( A \) and \( x = (x_B, x_N) \) be any feasible solution of LP. We have shown above that

\[ z(x) = c_B^T B^{-1} b - (c_B^T B^{-1} N - c_N^T) x_N. \]

A sufficient condition for the optimality of the basic solution with respect to the basis \( B \) is the following:

1) The basic solution which corresponds to the basis \( B \) is feasible.
2) \( c_B^T B^{-1} N - c_N^T \leq 0 \).

We shall investigate the situation that both properties held at the same time.

Let \( \hat{B} \) and \( \tilde{B} \) be two bases of \( A \) such that

\[
T \hat{B}^{-1} b \leq c_B^T \tilde{B}^{-1} b.
\]

Assume that the basic solution \((x_{\hat{B}}, x_N)\) is feasible for LP; i.e., property 1 holds with respect to the basic \( \hat{B} \).

\[
T \hat{B}^{-1} b = 2Q x = c_B^T \tilde{B}^{-1} b - (c_B^T \tilde{B}^{-1} N - c_N^T) x_N.
\]

Since \( x \) is a feasible solution for LP there holds

\[
x_N \geq 0.
\]

Hence

\[
T c_B^T \tilde{B}^{-1} b < c_B^T \tilde{B}^{-1} b
\]

\[
\Rightarrow
\]

\[
(c_B^T \tilde{B}^{-1} N - c_N^T) \neq 0
\]
Therefore, the basis $B$ cannot fulfill Property 2.

Hence, only a basis $B$ which maximizes

$$c_{iB}^T B^{-1} b$$

under fulfillment of

$$c_{iB}^T B^{-1} N \leq c_N^T$$

can solve the linear program $LP$.

**Question:**

Can we express this as an optimization problem?

Let $$\Pi^T := c_B^{-1} B^{-1}.$$ Then the following linear program defines such an optimization problem:

$$LP': \max \ w(\Pi) = \Pi^T b$$

$$\Pi^T A \leq c_N^T$$

$$\Pi \geq 0.$$ 

$\Pi \geq 0$ means that we have no restrictions on the variables in $\Pi$.

$LP$ is called the primal linear program and $LP'$ is the dual linear program of $LP$.

**Question:**

What are the relations of the feasible solutions of a primal linear program and its dual linear program?
Let $x$ and $\Pi$ be feasible solutions of the primal and the dual linear program. Then

$$\langle x, \Pi \rangle = c^T x \geq \Pi^T A x = \Pi^T b.$$  \hspace{2cm} (\star)

The relation \((\star)\) is called weak duality.

Weak duality says that the cost of the primal linear program is always at least as large as the cost of its dual linear program.

Hence the existence of a feasible solution of the primal linear program implies that the dual linear program cannot be unbounded.

Therefore, the existence of an optimal solution for the primal linear program implies the existence of an optimal solution for the dual linear program.

Note that with respect to an optimal basic solution $(x_B, x_N)$ of the primal linear program there holds

$$c_B B^{-1} N - c_N \leq 0$$

The reverse consideration holds as well.

We wish to prove that the cost of the optimal solutions of the primal and the dual linear programs are always equal.

Because of the weak duality it suffices to construct an explicit dual feasible solution
Such that $\Pi_0^T b = C x_0$ for an optimal solution $x_0$ of the primal linear program.

Since the primal linear program has an optimal solution, Theorem 3.3 implies the existence of a basis $\hat{B}$ of the matrix $A$ such that the corresponding basic solution $(x_{\hat{B}}, x_{\hat{B}^c})$ is also optimal; i.e., $z(x_0) = C_{\hat{B}}^T \hat{B}^{-1} b$.

Let $\Pi_0 := C_{\hat{B}}^T \hat{B}^{-1}$. Then

$$w(\Pi_0) = \Pi_0^T b = C_{\hat{B}}^T \hat{B}^{-1} b = C^T x_0 = z(x_0).$$

Altogether, we have proved the following theorem.

**Theorem 3.6**

If a linear program has an optimal solution, then its dual linear program has also an optimal solution. The cost of both optimal solutions are equal.

An important property of duality is its symmetry. This means that the dual of the dual linear program is the primal linear program again. To prove this property, it is useful to consider a linear program in general form and to determine its corresponding dual linear program. For doing this, we transform the general linear program to standard form such that we can...
apply the above construction. Consider

\[
\text{LP: } \min z(x) = c^T x \\
A_i x = b_i \quad i \in M \\
A_i x \geq b_i \quad i \in N \\
x_j = 0 \quad j \in N \\
x_j \leq 0 \quad j \in N
\]

where \( A_i \) denotes the \( i \)-th row of the matrix \( A \).

\text{Goal: The transformation of LP to standard form.}

We create

1) for each inequality \( A_i x \geq b_i \), \( i \in N \)
a slack variable \( x_i \)

and

2) for each unrestricted variable \( x_j \), \( j \in N \)
two new nonnegative variables \( x_j^+ \) and \( x_j^- \)
with \( x_j = x_j^+ - x_j^- \) and replace the column \( a_j \) by two columns \( a_j \) and \(-a_j\).

Then we obtain the following equivalent linear program

\[
\min z(x) = c^T x \\
\hat{A} x = \hat{b} \\
\hat{x} \geq 0
\]

where
\[
\hat{A} = \begin{bmatrix} a_j, j \in \mathbb{N} & | & (a_j, -a_j), j \in \mathbb{N} & | & 0, i \in \mathbb{M} \\
\end{bmatrix}
\]

\[
\hat{x} = (x_j, j \in \mathbb{N} | (x^+, x^-), j \in \mathbb{N} | x^s, i \in \mathbb{M})^T
\]

\[
\hat{c} = (c_j, j \in \mathbb{N} | (c_j, -c_j), j \in \mathbb{N} | 0)^T
\]

Then we obtain

\[
\hat{z} (\hat{x}) = \hat{c}^T \hat{B}^{-1} \hat{b} - (\hat{c}^T \hat{B}^{-1} \hat{N} - \hat{c}^T \hat{N}) \hat{x}^0
\]

Let

\[
\Pi^T := \hat{c}^T \hat{B}^{-1}.
\]

Then we obtain the following dual linear program:

\[
\begin{align*}
\max w(\Pi) &= \Pi^T \hat{b} \\
\Pi^T \hat{A} &\leq \hat{c}^T \\
\Pi &\leq 0.
\end{align*}
\]

\(\hat{A}\) and \(A\) have the same number of rows. Each inequality in \(\Pi^T \hat{A} \leq \hat{c}^T\) corresponds to the product of \(\Pi^T\) with a column in \(\hat{A}\).
The columns in \(\hat{A}\) are separated into three sets. This separates the inequalities in \(\Pi^T \hat{A} \leq \hat{c}^T\) as follows:

1. \(\Pi^T a_j \leq c_j\) for \(j \in \mathbb{N}\)
2. \(\Pi^T a_j \leq c_j\) and \(-\Pi^T a_j \leq -c_j\)

\(\Rightarrow \Pi^T a_j = c_j\) for \(j \in \mathbb{N}\) and
3. \(-\pi_i \leq 0 \iff \pi_i \geq 0\) for \(i \in M\)

Therefore, the dual linear program can be written in the following way:

\[
LP^* : \max \ w(\pi) = \pi^T b
\]

\[
\begin{align*}
\pi^T a_j &\leq c_j \quad j \in N \\
\pi^T a_j &= c_j \quad j \in N \\
\pi_i &\leq 0 \quad i \in M \\
\pi_i &\geq 0 \quad i \in M
\end{align*}
\]

Let the linear program \(LP\) in general form be the primal linear program. Then the dual linear program of \(LP\) is the linear program \(LP^*\):

<table>
<thead>
<tr>
<th>primal</th>
<th>dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\min \ z(x) = c^T x)</td>
<td>(\max \ w(\pi) = \pi^T b)</td>
</tr>
<tr>
<td>(Ax = b) \quad i \in M</td>
<td>(\pi_i \leq 0)</td>
</tr>
<tr>
<td>(Ax \geq b) \quad i \in M</td>
<td>(\pi_i \geq 0)</td>
</tr>
<tr>
<td>(x \geq 0) \quad j \in N</td>
<td>(\pi^T a_j \leq c_j)</td>
</tr>
<tr>
<td>(x \leq 0) \quad j \in N</td>
<td>(\pi^T a_j = c_j)</td>
</tr>
</tbody>
</table>

Now we can prove the symmetry property of the duality.

**Theorem 3.7**

The dual linear program of the dual is the primal linear program again.
Proof:

First we transform the dual linear program such that

- we have a minimizing problem,
- \( B \) is the only kind of inequalities in it,
- the rows of the matrix are multiplied with the column vector of the variables.

Then we shall use the resulting linear program as input for the transformation described above.

\[
\begin{align*}
\max \ w(\pi) &= \pi^T b \\
-\pi^T a_j &\leq c_j & j \in N \\
\pi^T a_j &= c_j & j \in \bar{N} \\
\pi_i^T &\geq 0 & i \in \bar{M} \\
\pi_i^T &\leq 0 & i \in M \\
\end{align*}
\]

\[
\begin{align*}
\min \ w(\pi) &= \pi^T (-c) \\
-(-a_j^T) \pi &\geq -c \\
(-a_j^T) \pi &= -c \\
\pi_i^T &\geq 0 \\
\pi_i^T &\leq 0 \\
\end{align*}
\]

After the application of the construction above, we obtain the following dual linear program.

\[
\begin{align*}
\max \ z(x) &= x^T (-c) \\
x_j &\geq 0 & j \in N \\
x_j &= 0 & j \in \bar{N} \\
-A_i x &\leq -b_i & i \in \bar{M} \\
-A_i x &= -b_i & i \in M \\
\end{align*}
\]

This is equivalent to the initial primal linear program.
\[ \begin{align*}
\min z(x) &= c^T x \\
A_i x &= b_i \quad \text{if } i \in M \\
A_i x &\geq b_i \quad \text{if } i \in M \\
x_j &\geq 0 \\
x_j &\leq 0 \\
\end{align*} \]

For each linear program \( LP \) exactly one of the following three cases is fulfilled:

1) \( LP \) has a finite optimum.
2) \( LP \) is unbounded.
3) \( LP \) has no feasible solution.

The following table describes which combinations of the three cases above are possible for a primal and its dual linear program.

<table>
<thead>
<tr>
<th>primal/dual</th>
<th>finite optimum</th>
<th>unbounded</th>
<th>infeasible</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite optimum</td>
<td>1</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>unbounded</td>
<td>x</td>
<td>x</td>
<td>3</td>
</tr>
<tr>
<td>infeasible</td>
<td></td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Theorems 3.6 and 3.7 exclude in the first row and the first column of the table all cases up to combination 1. If the primal or the dual linear program has unbounded cost, the other linear program cannot have a feasible solution.
To show that the combinations 2 and 3 occur, we give simple examples. Consider

LP: \[ \text{min} \ z(x) = x_1 \]
\[ x_1 + x_2 \geq 1 \]
\[ -x_1 - x_2 \geq 1 \]
\[ x_1 \leq 0 \]
\[ x_2 \leq 0 \]

Obviously, LP has no feasible solution. Its dual linear program is

LP': \[ \text{max} \ w(\pi) = \pi_1 + \pi_2 \]
\[ \pi_1 - \pi_2 = 1 \]
\[ \pi_1 - \pi_2 = 0 \]
\[ \pi_1 \geq 0 \]
\[ \pi_2 \geq 0 \]

LP' has also no feasible solution.

⇒

We have an example for combination 2.

If we replace in LP the restrictions of the variables \( x_1 \) and \( x_2 \) by \( x_1 \geq 0 \) and \( x_2 \geq 0 \) then the primal linear program remains to be infeasible but its dual linear program gets to be unbounded.

Altogether, we have proved the following theorem.
**Theorem 3.8**

A primal-dual pair of linear programs always fulfills exactly one of the combinations 1-3 as described in the above table.

Let us consider the primal linear program in general form and its dual linear program again. The stronger restrictions of the primal linear program (dual linear program) correspond to the sets $M$ and $N$ ($\overline{M}$ and $\overline{N}$) of indices.

A necessary and sufficient condition of a pair $x, \overline{x}$ of feasible solutions for a primal-dual pair is the so-called complementary slackness condition.

**Theorem 3.9 (complementary slackness)**

Let $x, \overline{x}$ be feasible solutions for a primal-dual pair of linear programs. Then $x$ and $\overline{x}$ are optimal iff

1) $u_i = \overline{v}_i (A_i x - b_i) = 0$ for all $i \in M \cup \overline{M}$ and
2) $v_j = (c_j - \overline{u}_j a_j^T) x_j = 0$ for all $j \in N \cup \overline{N}$.

**Proof**:

The restrictions of both linear programs imply

$u_i \geq 0$ for all $i \in M \cup \overline{M}$ and

$v_j \geq 0$ for all $j \in N \cup \overline{N}$.
Let
\[ u := \sum_{i \in \mathcal{M} \cup \mathcal{N}} u_i \quad \text{and} \quad v := \sum_{j \in \mathcal{N} \cup \mathcal{N}} v_j. \]

Then \( u \geq 0 \) and \( v \geq 0 \).

Furthermore
\[
\begin{align*}
  u = 0 & \iff \sum_{i \in \mathcal{M} \cup \mathcal{N}} (A_i x - b_i) = 0 \quad \text{for all } i \in \mathcal{M} \cup \mathcal{N} \\
  v = 0 & \iff \sum_{j \in \mathcal{N} \cup \mathcal{N}} (c_j - \Pi^T a_j) x_j = 0 \quad \text{for all } j \in \mathcal{N} \cup \mathcal{N}.
\end{align*}
\]

Consider
\[
\begin{align*}
  u + v &= \sum_{i \in \mathcal{M} \cup \mathcal{N}} (A_i x - b_i) + \sum_{j \in \mathcal{N} \cup \mathcal{N}} (c_j - \Pi^T a_j) x_j \\
  &= C^T x - \Pi^T b.
\end{align*}
\]

\[ \Rightarrow \]
\[
\begin{align*}
  \left[ (u_i = 0 \quad \forall i \in \mathcal{M} \cup \mathcal{N}) \quad \text{and} \quad (v_j = 0 \quad \forall j \in \mathcal{N} \cup \mathcal{N}) \right] \\
  \iff \\
  u + v = 0 \\
  \iff \\
  C^T x = \Pi^T b.
\end{align*}
\]

Thus for feasible solutions \( x \) and \( \Pi \) always
\[ C^T x \geq \Pi^T b. \]

Theorem 3.6 imply the assertion.
We have developed the simplex algorithm for a given primal linear program. This algorithm is often called primal simplex algorithm.

The primal simplex algorithm solves the linear program by going from a feasible basic solution of the primal linear program (or shortly a primal feasible basis) to another neighbour feasible basic solution.

Goal:

The development of a simplex algorithm which solves the linear program by going from a dual feasible basic solution to another neighbour dual feasible basis.

Such an algorithm is called dual simplex algorithm.

Let us consider a primal-dual pair where the primal linear program is in standard form again. Let

\[ z_0 := \min \left\{ c^T x \mid A x = b, \ x \geq 0 \right\} \quad \text{and} \quad w_0 := \max \left\{ \frac{1}{t}^T b \mid t^T A \leq c^T, \ t \geq 0 \right\}, \]

be the cost of an optimal solution of the primal and dual linear program, respectively.

Let \( B \) be any basis of \( A \).
The corresponding basic solution \((x_B, x_N)\) with \(x_B = B^{-1}b\) and \(x_N\) is feasible iff \(B^{-1}b \geq 0\). Then \((x_B, x_N)\) is a primal feasible basic solution.

\(\Pi^T = c_B^T B^{-1}\) is the solution corresponding to the dual linear program with respect to the basis \(B\).

We shall investigate the situations when \(\Pi^T = c_B^T B^{-1}\) is feasible for the dual linear program.

Consider \(\Pi^T A - c^T\). Then we obtain

\[
\Pi^T A - c^T = c_B^T B^{-1}(B^T N) - (c_B, c_N)^T
= (0, c_B^T B^{-1} N - c_N)^T.
\]

Therefore, \(\Pi^T = c_B^T B^{-1}\) is feasible for the dual linear program iff

\[c_B^T B^{-1} N - c_N \leq 0\]

A basis \(B\) of the matrix \(A\) is called dual feasible if \(c_B^T B^{-1} N \leq c_N^T\).

We have seen that a basis \(B\) defines the point

\[x = (x_B, x_N) = (B^{-1}b, 0) \in \mathbb{R}^n\]

and the point

\[\Pi^T = c_B^T B^{-1} \in \mathbb{R}^m.\]
If a basis $B$ of $A$ is primal and dual feasible then $x = (x_B, x_N) = (B^{-1}b, 0)$ and $\pi^* = c_B^T B^{-1}$ are optimal solutions of the primal and the dual linear program.

**Proof:**

Since $c^T x = c_B^T B^{-1} b$ and $\pi^T b = c_B^T B^{-1} b$ and hence $c^T x = \pi^T b$, the assertion follows.

**Interpretation:** The

- **primal simplex algorithm** starts with a primal feasible basis and tries to get dual feasibility while maintaining primal feasibility.

- **dual simplex algorithm** starts with a dual feasible basis and tries to get primal feasibility while maintaining dual feasibility.

We could solve the dual linear program in the following way:
(1) Transform the linear program to standard form such that the primal simplex can be applied.

(2) Use the primal simplex algorithm to solve the dual linear program.

Instead of doing this, we shall develop a direct algorithm for the solution of the dual linear program.

**Exercise:**

Given the dual of a primal linear program, transform the dual linear program such that the primal simplex algorithm could be applied.

Let $B$ be a dual feasible basis of $A$. Since the dual linear program is feasible, the following combinations for the primal-dual pair of linear programs are possible:

<table>
<thead>
<tr>
<th>Primal/dual</th>
<th>finite optimum</th>
<th>unbounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite optimum</td>
<td>1</td>
<td>x</td>
</tr>
<tr>
<td>infeasible</td>
<td>x</td>
<td>3</td>
</tr>
</tbody>
</table>

$\Rightarrow$

The dual simplex algorithm can terminate with the following results.
1) A finite optimum is found.
2) It is established that the primal linear program is infeasible.

Hence it is useful to develop a sufficient criterion of the infeasibility of the primal linear program.

To get such a criterion let

\[ x = (x_B, x_N) \]

be a feasible solution of the primal linear program. Then

\[ Bx_B + N x_N = b \] implies

\[ x_B + B^{-1} N x_N = B^{-1} b. \] \( (*) \)

Let \( R \) be the set of indices of the nonbasic variables. Then, we can write \((*)\) as follows.

\[ x_B + \sum_{j \in R} B^{-1} a_j x_j = B^{-1} b. \]

Let

\[ B^{-1} b = (y_1, y_2, \ldots, y_m)^T. \]

If the basis \( B \) is also primal feasible; i.e.,

\[ y_i \geq 0 \] for all \( 1 \leq i \leq m \). Lemma 3.2 implies that

\[ (B^{-1} b, 0) \] and \( c_B^T B^{-1} \)

are optimal solutions for the primal and dual
Assume that the basis $B$ is not primal feasible and let $B_1', B_2', \ldots, B_m'$ be the rows of $B^{-1}$.

Then $\exists \ p \in \{1, 2, \ldots, m\}$ with $y_{p0} < 0$. The following lemma gives us the desired criterion.

**Lemma 3.3**

Let $B$ be a dual feasible basis and let $B^{-1} b = (y_{10}, y_{20}, \ldots, y_{m0})^T$. Let $y_{p0} < 0$. If $B_p a_j > 0$ for all $j \in R$ then the primal linear program is infeasible.

**Proof:**

Note that

$$y_{p0} = x_{B_p} + \sum_{j \in R} B_p a_j x_j,$$

By definition, for each feasible primal solution there hold $x_j \geq 0$ for all $j \in R$.

Since $y_{p0} < 0$ and $B_p a_j > 0$ for all $j \in R$ there has to be

$$x_{B_p} < 0.$$

$\Rightarrow$

The primal linear program has to be infeasible.
If the criterion of Lemma 3.3 is not fulfilled then there is $q \in \mathbb{R}$ with $B_q \vec{a}_q \leq 0$.

Idea

Determine such a $q$ and perform the basis exchange $a_{B_p} \leftrightarrow a_q$.

We have to take care that the dual feasibility is not destroyed by this exchange.

$\Rightarrow$

For $j \in \mathbb{R}$ there holds $y_{o_j} = -c_j$.

Dual feasibility of $B' \Rightarrow$

$y_{o_j} \leq 0$ for all $j \in \mathbb{R}$.

After the basis exchange $a_{B_p} \leftrightarrow a_q$, we obtain the following value $y'_{o_j}$ for $-c_j$:

\[
y'_{o_j} = y_{o_j} - \frac{y_{o_q}}{y_{p_q}} y_{p_j}
\]

Since the new basis $B'$ has to be dual feasible, we have to take care that for all $j \in \mathbb{R}' = \{B_p\} \cup R \setminus B_q$

$y'_{o_j} \leq 0$

$\iff y_{o_j} - \frac{y_{o_q}}{y_{p_q}} y_{p_j} \leq 0$. (1)
We distinguish three cases.

a) $y_{pj} = 0$:

Then $y_{oj} \leq 0$ implies that the inequality (+) is fulfilled.

b) $y_{pj} > 0$:

Since $y_{pq} = B_{p}^{T} C_{q} < 0$ and $y_{oj} \leq 0$
the inequality (+) is fulfilled as well.

c) $y_{pj} < 0$:

Then we have to take care. We obtain

$$y_{oj} - \frac{y_{oj}}{y_{pq}} y_{pj} \leq 0$$

$\Rightarrow$

$$\frac{y_{oj}}{y_{pj}} \leq \frac{y_{oj}}{y_{pq}}$$

If we choose $q$ such that

$$\frac{y_{oj}}{y_{pq}} = \max \left\{ \frac{y_{oj}}{y_{pj}} \mid y_{pj} < 0 \right\}$$

then the inequality (+) will be always fulfilled.

**Exercise:**

Work out in detail the dual simplex algorithm. Could you use the same tables as the primal simplex algorithm?