On the Computational Complexity of Quantified Horn Clauses

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Abstract

A polynomial time algorithm is presented for the evaluation problem for quantified propositional Horn clauses. This answers an open problem of [IM 87].
1 Introduction

The basic idea behind the programming language PROLOG is, that a proof or refutation of Horn formulas can be viewed as an efficient computation from which one extracts an output. Horn formulas (or a program) are conjunctions of Horn clauses, i.e. clauses of the form: $A_1 \land A_2 \land \ldots \land A_m \rightarrow B$, where $A_i$ and $B$ are atomic formulas. The computation of the program consists in finding an assignment of values to the variables which satisfies all clauses. Two basic methods used for such a computation are unification (to produce assignments) and resolution of clauses (as a method of logical inference). The problem of testing a set of Horn formulas for satisfiability, e.g. using unit resolution is known to have linear time solution algorithms, see e.g. [DG 84], [IM 87].

The new development of Prolog query languages, cf. [GR 87], strongly motivated the search for general efficient solutions for quantified Horn clauses. In this paper we present such an efficient solution for the evaluation problem of quantified propositional Horn clauses. The algorithm works in $O(n^3)$ time. We stress in this paper polynomial time solutions for this problem, rather than the design of new data structures to make it work faster. We do hope though to present a linear time or $O(n \log n)$ time algorithm in a subsequent paper. Schäfer in [Sch 78] claimed a polynomial time algorithm for the above problem, but he did not give a proof.

It is interesting to note, that the evaluation problem for quantified Boolean formulas, even if restricted to formulas containing at most three literals per clause, is PSPACE-complete, [GJ 79]. In [APT 79] a linear time algorithm has been design for the case of quantified Boolean formulas in conjunctive normal form with at most two literals per clause.

Our result in this context entails suprising algorithmic efficiency of the evaluation problem for quantified Horn clauses and opens the possibility of several natural query-like extensions of standard PROLOG.

2 Terminology

We will mainly deal with propositional and quantified propositional formulas. We denote by $\text{PV}$ the infinite set of propositional variables. The propositional connectives “and”, “or”, and “not” are designated by the symbols ”$\land$”, ”$\lor$”, and ”$\lnot$”. We shall have occasion to deal with the following special sets of propositional formulas: literals, all propositional formulas, clauses, conjunctive normal forms (conjunctions of clauses), Horn clauses (clauses containing at most one positive literal), totally negative clauses (clauses containing only negative literals), negative conjunctive normal forms (conjunctions of totally negative clauses). These will be denoted respectively by $\text{LI}$, $\text{FML}$, $\text{CL}$, $\text{CNF}$, $\text{HC}$, $\text{TNC}$, $\text{NCNF}$.

In case we allow the propositional constants 0 and 1 to occur, the corresponding sets are indexed by $C$, e.g. $\text{CNF}_C$, $\text{HC}_C$, etc.

The universal (existential) quantifier will be denoted by $\forall$ ($\exists$) as usual. For any set $\Sigma$ of propositional formulas $Q^*\Sigma$ is the set of all formulas of the form $Q_1X_1\ldots Q_nX_n$, where
n is an arbitrary natural number, each $Q_i$ is either $\forall$ or $\exists$, $o \in \Sigma$ and $\{X_1, \ldots, X_n\}$ is a set of propositional variables. If $\{X_1, \ldots, X_n\}$ contains all variables occurring in $o$, we call $Q_1 X_1 \ldots Q_n X_n$ closed. Likewise $\exists^* \Sigma$ is the set of all formulas of the form $\exists X_1, \ldots, \exists X_n$, with $o \in \Sigma$.

As usual we distinguish free and bound occurrences of variables in quantified propositional formulas. For $X \in PV$ and $t$ a constant symbol the formula $o(t/X)$ arises from $o$ by replacing every free occurrence of $X$ in $o$ by $t$.

We say that a variable $X$ occurs in a clause $\phi$ if either $X$ or $\neg X$ is a disjunctive component of $\phi$. In contrast we say that $X$ is a literal of $\phi$ if $X$ is a disjunctive component of $\phi$. Thus $X$ occurs in $\neg X \lor Y \lor \neg Z$, but $X$ is not a literal of this clause.

### 3 Generalized Unit Resolution

It is well known that a formula of the form

$$\exists X_1 \ldots \exists X_k \phi$$

where $\phi$ is a conjunction of Horn clauses, is true if and only if the empty clause is not derivable from $\phi$ by unit resolution. We first generalize the operation of unit resolution to the case of arbitrary quantifier prefixes.

Let $\Phi$ be a formula of the form

$$\forall \vec{X}_1 \exists Y_1 \ldots \exists Y_{k-1} \forall \vec{X}_{k\phi}$$

where

- for all $i, 1 \leq i \leq k$ \quad $\vec{X}_i = X_{n_{i-1}+1}, \ldots, X_{n_i}$, with $n_0 = 0$
- for all $i, 1 \leq i < k$ \quad $\vec{Y}_i = Y_{m_{i-1}+1}, \ldots, Y_{m_i}$, where all $m_i \neq 0$.
- $\phi = \phi_1 \land \ldots \land \phi_r$
- where all $\phi_i$ are Horn clauses:

we may furthermore assume without loss of generality:

- for every $i, 1 \leq i \leq r$, there is no variable $X_j$, such that both literals $X_j$ and $\neg X_j$ occur in $\phi_i$

In the given representation of $\Phi$ we assume implicitly, that no propositional variable occurs both existentially and universally bound. This is clearly no restriction.

An **X-literal** (**Y-literal**) is a literal of the form $X_i$ or $\neg X_i$ (resp. $Y_i$ or $\neg Y_i$). A **pure X-clause** is a clause consisting exclusively of $X$-literals. In particular the empty clause is a pure $X$-clause.
A clause $\phi_j$ is called a $Y_i$-unit clause if $Y_i$ occurs positively in $\phi_j$ and $Y_i$ is the only $Y$-variable occurring in $\phi_j$.

A clause $\phi_j$ is called a $Y$-unit clause if $Y_i$-unit clause for some $i$.

When we say that the variable $X_i$ is before $Y_j$, we refer to the order of occurrences in the prefix of $\Phi$, i.e., $X_i$ is before $Y_j$ if $n_{p-1} < i \leq n_p, m_{q-1} < j \leq m_q$ and $p \leq q$. Analogously we use the phrase $X_i$ is after $Y_j$.

Let $\phi_p$ be a $Y_i$-unit clause and $\phi_q$ a clause containing the literal $-Y_i$. The resolvent $\psi$ of $\phi_p$ and $\phi_q$ is obtained by

forming the disjunction $\phi_p \lor \phi_q$,

if for some variable $X_j$ both literals $X_j$ and $-X_j$ occur in $\phi_p \lor \phi_q$, then stop.

No resolvent exists in this case.

omitting all occurrences both negated and unnegated of the variable $Y_i$,

omitting all occurrences of $X$-variables, that are not before any $Y$-variable occurring in the modified disjunction,

A unit resolution step on the formula $\phi$

$$\forall X_1 \exists Y_1 \ldots \exists Y_k \Phi = 1 \forall X_k (\phi_1 \land \ldots \land \phi_k)$$

is performed by adding the resolvent $\psi$ of a $Y$-unit clause $\phi_p$ and a clause $\phi_q$ containing the literal $-Y$ to the matrix of $\phi$, thus obtaining

$$\forall X_1 \exists Y_1 \ldots \exists Y_{k-1} \forall X_k (\phi_1 \land \ldots \land \phi_k \land \psi)$$

**Lemma 1:** Let $\sum$ be obtained from $\Phi = \forall X_1 \exists Y_1 \ldots \exists Y_{k-1} \forall X_k \phi$ by omitting in any clause of $\phi$ all $X$-literals for those $X_i$ that are not before any $Y$-variable occurring in this clause, then $\sum$ is true if and only if $\phi$ is true.

**Proof:** If $\sum$ is true, then $\phi$ is, of course, also true. So let us assume that $\Phi$ is true. There are functions $f_i$ for all $i$, $1 \leq i \leq m_{k-1}$, the number of arguments of $f_i$ equals the number of $X$-variables that are before $Y_j$, such that for any sequence $a_1, \ldots, a_{n_k}$ of $0$ and $1$ the formula

$$\phi(a_1, \ldots, a_{n_k}, f_1(a^1), \ldots, f_{m_{k-1}}(a^{m_{k-1}}))$$

is true, where $b_i = f_i(a_1, \ldots, a_n)$ for the appropriate number $n$ of arguments.

Let $\phi_i$ be a clause in the matrix of $\Phi$. Let $\phi_i = \phi_{i,1} \lor \phi_{i,2}$, where $\phi_{i,1}$ contains all $Y$-literals from $\phi_i$ and those $X$-literals of $\phi_i$, such that $X$ occurs before $Y$ for some $Y$-literal in $\phi_i$ and $\phi_{i,2}$ contains all literals $-X_j$ or $X_j$ such that $X_j$ is not before any $Y$-variable occurring in $\phi_i$.

Let us fix an assignment $a_1, \ldots, a_{n_k}$ of $0$ and $1$. We will use $b_i$ as an abbreviation for $f_i(a^1)$.

We will show, that

$$\phi_{i,1}(a_1, \ldots, a_{n_k}, f_1(a^1), \ldots, f_{m_{k-1}}(a^{m_{k-1}}))$$
is true.

Let \( a'_1, \ldots, a'_n \) be another assignment of values 0 and 1 to the \( X \)-variables which agrees with the fixed assignment except possibly for variables \( X_j \) that are not before any \( Y \)-variable in \( \phi_i \). The assignment \( \tilde{a}' \) is chosen to have the property, that \( \phi_{i,2}(\tilde{a}', \tilde{b}') \) is false. This is possible since by assumption on \( \Phi \) the pure \( X \)-clause \( \phi_{i,2} \) contains no complementary pair \( X_j, \tilde{X}_j \).

Since \( \phi \) is true under the assignment \( a'_1, \ldots, a'_n, b'_1, \ldots, b'_{m-1} \) \( \phi_{i,1} \) has to be true. Since \( \phi_{i,1} \) contains only \( X \)-variables \( X_i \) for which \( a_i = a'_i \) and since for all variables \( Y_j \) in \( \phi_{i,1} \) the function \( f_j \) does not have any of the changed \( X \)-values among it arguments, \( \phi_{p,1} \) is also true under the original assignment \( \tilde{a} \).

**Lemma 2:** Let \( \Sigma \) be obtained from \( \Phi \) by a resolution step, then \( \Sigma \) is true if only if \( \Phi \) is true.

**Proof:** If \( \Sigma \) is true, then \( \Phi \) is, of course, also true. To prove the converse direction we first observe, that adding to \( \phi \) the ordinary resolvent of two clauses, without omitting any variables leads to a logically equivalent formula \( \phi' \). Now Lemma 1 is used to pass from \( \forall \tilde{X}_1 \exists \tilde{Y}_1 \ldots \exists \tilde{Y}_{k-1} \forall \tilde{X}_k \phi' \) to \( \Sigma \).

**Theorem 3:** Let \( \Phi \) be a formula of the form \( \forall \tilde{X}_1 \exists \tilde{Y}_1 \ldots \exists \tilde{Y}_{k-1} \forall \tilde{X}_k \phi \). \( \Phi \) is true if and only of no pure \( X \)-clause can be derived from \( \phi \) by \( Y \)-unit resolution.

**Proof:** Let \( \Phi' \) be obtained from \( \Phi \) by \( Y \)-unit resolution, such that the matrix \( \phi' \) of \( \Phi' \) contains a pure \( X \)-clause \( \psi \). By assumption on \( \Phi \) and the definition of unit resolution \( \psi \) cannot contain a complementary pair. Thus \( \Phi' \) is obviously false. By Lemma 1 also \( \Phi \) has to be false.

Now let us assume, that no pure \( X \)-clause can be derived from \( \phi \). Let \( \sigma \) be the conjunction of \( \phi \) together with all resolvents, that can be derived by \( Y \)-unit resolution and let \( \Sigma \) be the formula with the same prefix as \( \Phi \) and the matrix \( \sigma \). We will show that \( \Sigma \) is true, which immediately yields also the truth of \( \Phi \).

Let an arbitrary assignment \( \tilde{a} = a_1, \ldots, a_n \) of 0 and 1 be given. We define the assignments \( \tilde{b}_i \) for the variables \( Y_i \) as follows:

- If there is a \( Y_i \)-unit clause in \( \sigma \) that is not already made true under the partial assignment \( \tilde{a} \), then let \( \tilde{b}_i \) equal 1.
- Let \( \tilde{b}_i \) equal 0 otherwise.

We prove by induction on the number \( s \) of \( Y \)-variables in \( \chi \), that \( \chi(\tilde{a}, \tilde{b}) \) is true, for any clause \( \chi \) of \( \sigma \).

If \( s = 0 \), then \( \chi \) would consist entirely of \( X \)-variables. By assumption this is not possible.
Let \( s = 1 \). If \( \chi \) is a \( Y_i \)-unit clause, then \( \chi \) is either already true on the basis of the assignment \( \vec{a} \) or \( b_i \) has been defined to be equal to 1. So let us assume that the only \( Y_i \)-literal in \( \chi \) is \( \neg Y_i \). If no \( Y_i \)-unit clause occurs in \( \phi \), that is not true on the basis of \( \vec{a} \) alone, then \( b_i \) is equal to 0 and \( \chi \) is true. Finally it remains to consider the case that \( \phi \) contains a \( Y_i \)-unit clause \( \chi' \), that is not already true under the partial assignment \( \vec{a} \). By the Horn property all \( X_j \)-literals in \( \chi' \) are negative. We may therefore draw the conclusion that for all variables \( X_j \) in \( \chi' \) \( a_j = 1 \). By assumption \( \phi \) and \( \phi' \) cannot have a resolvent, i.e. for some \( j \) \( X_j \) occurs in \( \chi \) and \( \neg X_j \) occurs in \( \chi' \), which implies that \( \chi \) is true, since \( a_j = 1 \).

Induction step. Let \( \chi \) contains \( s + 1 \) \( Y \)-literals, where we may now assume \( s > 1 \). By the Horn property \( \chi \) has to contain a negative \( Y \)-literal \( \neg Y_i \). If there is no \( Y_i \)-unit clause in \( \phi \), then \( b_i = 0 \) and \( \chi \) is true. Otherwise let \( \chi' \) be such a clause. The resolvent \( \psi \) of \( \chi \) and \( \chi' \) contains \( s \) \( Y \)-literals and is thus true by induction hypothesis. Since the disjunctive part of \( \psi \) stemming from \( \chi' \) is not true, the part stemming from \( \chi \) has to be. Thus also \( \chi \) is true.

4 Examples

**Example 1** Let
\[
\Phi = \forall X \exists Y ((X \lor \neg Y) \land (\neg X \lor Y))
\]
The second clause is a \( Y \)-unit clause. Its resolvent with the first clause would contain the complementary pair \( X \) and \( \neg X \) and is thus not performed. No pure \( X \)-clause is derivable, \( \Phi \) is true.

**Example 2** Let
\[
\Phi = \forall Y \exists X ((X \lor \neg Y) \land (\neg X \lor Y))
\]
Again the second clause is a \( Y \)-unit clause. It's resolvent with the first clause is the empty clause since all occurrences of \( X \)-variables are dropped. Thus the formula is false.

**Example 3** Let
\[
\Phi = \exists Y_i \forall X \exists Y_2 ((\neg Y_i \lor X \lor \neg Y_2) \land (\neg Y_1 \lor \neg X \lor Y_2) \land Y_1)
\]
The only \( Y \)-unit clause is \( Y \). The resolvents with the first and second clause are \( X \lor \neg Y_2 \) and \( \neg X \lor Y_2 \) respectively.

The second clause is again a \( Y \)-unit clause. No resolution with the first clause is possible, because a complementary pair would arise. This \( \Phi \) is true.

**Example 4** Let
\[
\Phi = \forall X_1 \forall X_2 \exists Y_1 \exists Y_2 ((X_2 \lor \neg Y_2) \land (Y_2 \lor \neg Y_1) \land (\neg X_2 \lor Y_1) \land (\neg X_1 \lor Y_1))
\]
Using the $Y$-unit clauses $\neg X_2 \lor Y_1$ and $\neg X_1 \lor Y_1$ we obtain by resolution with the second clause two new $Y$-unit clauses $Y_1 \lor \neg X_2$ and $Y_2 \lor \neg X_1$. Only the second of these can be used to continue resolution with the first clause to obtain $X_3 \lor \neg X_2$. Thus is false.

**Lemma 4:** Let $\Phi$ be a formula of the form $\forall X \exists Y \phi$. Then $\Phi$ is false if and only if for some assignment $\bar{a} = a_1, \ldots, a_n$ of values 0 and 1 with at most one occurrence of 0, the formula $\exists Y \phi(a_1, \ldots, a_n)$, is false.

**Proof:** One implication of the lemma is trivial. So let us assume that $\forall X \exists Y \phi$ is false. Let $\phi_0$ be the conjunction of $\phi$ together with all clauses, that can be derived from $\phi$ by $Y$-unit resolution. By Lemma 1 $\forall X \exists Y \phi$, is equivalent to $\forall X \exists Y \phi_0$. By our assumption the latter formula is false and thus contains by virtue of Theorem 3 a pure $X$-clause $\chi$. Let $\bar{a} = a_1, \ldots, a_n$ be an assignment of values 0 and 1 to the $X$-variables, such that $\chi(a_1, \ldots, a_n)$ is false. Since $\chi$ is a Horn clause, we may choose $\bar{a}$, such that at most one 0 occurs. Obviously $\exists Y \phi_0(a_1, \ldots, a_n)$ is false and therefore by the equivalence stated above also $\exists Y \phi(a_1, \ldots, a_n)$ is false.

**Lemma 5:** The truth of an $\forall \exists$-quantified conjunction of Horn formulas can be decided in polynomial time.

**Proof:** Let $\Phi = \forall X \exists Y \phi$. The algorithm consists in all assignments $\bar{a}$ with at most once 0 the truth of $\exists Y \phi(\bar{a})$. There are (number of $X$-variables) + 1 many assignments $\bar{a}$ with at most one 0. The reduction of $\phi(\bar{a})$ to a conjunction $\phi_1$ of Horn clauses not containing the constants 0 and 1 can be affected in linear time. Finally the satisfiability of $\phi_1$ can be decided in linear time. Thus the overall running time of the algorithm may be bounded by $(\text{length of input})^2$.

## 5 An Algorithm

Let $\Phi$ be a formula of the form

$$\forall X_1 \exists Y_1 \ldots \exists Y_{k-1} \forall X_k \phi$$

be given.

Let $N_\phi$ be the set of clauses in $\phi$, that are not pure $X$-clauses and contain only negative $Y$-literals. Let $P_\phi$ be the set of clauses in $\phi$, that contain at least one positive $Y$-literal.
for all clauses $C$ in $N_\phi$ do

let $S_C$ be the set of positive $X$-literals in $C$. {Thus $SC$ may be a singleton set or the empty set}

if $S_C$ is empty then do

\hspace{1em} remove all occurrences of all $X$-literals in $P_\phi$ and $C$ obtaining $P'_\phi$ and $C'$.
\hspace{1em} apply standard unit-resolution to $P'_\phi$ and $C'$.
\hspace{1em} if the empty clause can be derived terminate with “$\Phi$ is false”
\hspace{1em} otherwise terminate with “$\phi$ is true”.

end if

if $S_C = \{X_r\}$ then do

\hspace{1em} for all variables $X$ different from $X_r$ remove all $X$-literals from $P_\phi$
\hspace{1em} obtaining $P'_\phi$.
\hspace{1em} begin 1

\hspace{2em} let $L$ be the set of $Y$-unit clauses that may be derived from $P'_\phi$ by
\hspace{2em} standard unit resolution without talking the obstacle $X_r$ into
\hspace{2em} consideration and such that $Y$ occurs before $X_r$ in the prefix of $\Phi$.
\hspace{2em} for all $Y \in L$ remove all occurrences of the literal $\neg Y$ in all clauses
\hspace{2em} in $P'_\phi$.
\hspace{2em} remove all clauses containing a $Y$-literal, with $Y$ occurring before
\hspace{2em} $X_r$ in the prefix of $\Phi$ obtaining $P''_\phi$.
\hspace{2em} let $U$ be the set of $Y$-unit clauses in $P''_\phi$ not containing the literal $\neg X_r$.
\hspace{2em} let $R$ be the empty set.
\hspace{2em} while $U$ is not empty do

\hspace{3em} for $Y \in U$ do

\hspace{4em} remove all occurrences of the literal $\neg Y$ in all clauses in $P''_\phi$.
\hspace{4em} remove all clauses in $P''_\phi$ containing the literal $Y$.
\hspace{4em} add new $Y$-unit clauses not containing $\neg X_r$ to $U$.
\hspace{4em} remove $Y$ from $U$.
\hspace{4em} add $Y$ to $R$.

\hspace{3em} end for

\hspace{2em} end while

end 1

if all $Y$-variables in $C$ occur among the variables in $L \cup R$, then is “$\Phi$ is false”
otherwise “$\Phi$ is true”.

end if

end for

The complexity of the above algorithm is $O(n^3)$ observing that unit-resolution can be
performed in linear time.
References


