Lower Bound for Randomized Linear Decision Tree Recognizing a Union of Hyperplanes in a Generic Position

Dima Grigoriev* Marek Karpinski†

Abstract

Let $L$ be a union of hyperplanes with $s$ vertices. We prove that the runtime of a probabilistic linear search tree recognizing membership to $L$ is at least $\Omega(\log s)$, provided that $L$ satisfies a certain condition which could be treated as a generic position. A more general statement, namely without the condition, was claimed by F. Meyer auf der Heide [1], but the proof contained a mistake.
1 Families of hyperplanes in a generic position

Let $L = \bigcup_{1 \leq i \leq m} H_i \subset \mathbb{R}^k$ be a union of hyperplanes. We intend to define a version of what does it mean that $L$ is in a generic position.

If $Q = \bigcap_{1 \leq j \leq l} H_{i_j}$ has the dimension $\dim Q = l$ we call $Q$ $l$-face of $L$. Also $0$-faces we call vertices. If a hyperplane $H$ contains some $l$-face for rather $l$ then $H$ contains many vertices of $L$. The generic position for $L$ means, informally speaking, that this is the only reason for $H$ to contain many vertices of $L$.

**Definition.** We say that $L$ is in a generic position if for some $c_1 > c_2 > 0$, $c_3 > 0$ and any hyperplane $H \subset \mathbb{R}^k$

1) $L$ has $s \geq m^{c_1 n}$ vertices;

2) each vertex belongs to exactly $n$ hyperplanes of $L$.

3) the number of vertices $v$ lying in $H$ for which there is no $l$-face contained in $H$ such that this $l$-face contains $v$, where $l \geq c_3 n$, does not exceed $m^{c_2 n}$.

One can show that if $H_1, \ldots, H_m$ satisfy the property of algebraically independence, namely, that $m \cdot n$ coefficients $a_{ij}$ of all linear equations for $H_1, \ldots, H_m$ (i.e. $H_i = \{ \sum_{1 \leq j \leq n} a_{ij} x_j = 1 \}$) are algebraically independent over $Q$ then $L$ is in a generic position.

Moreover, one can prove in this case the following. Let $Q_1, \ldots, Q_t$ be all maximal (in the sense of inclusion) faces of $L$ contains in $H$, then $\sum_{1 \leq i \leq t} \dim(Q_i + 1) \leq n$. Thus, the number of vertices in the item considered in the item 3 of the definition does not exceed $n \cdot m^{c_2 n}$ since any $l$-face cannot contain more than $m^l$ vertices of $L$.

Let $D$ be a probabilistic linear search algorithm (or briefly $\alpha$-PLSA) recognizing $L$ with two-sided error $\alpha < 1/2$ (one can find in [1], [2] the concepts used in the present paper).

**Theorem.** If $L$ is in a generic position then the runtime of $D$ is greater than $\Omega(n \log m)$.

Note that the similar result was claimed in [1] even without the condition 3) from the definition of a generic position, but the proof contained a mistake.

For a value of the random parameter $0 \leq \gamma \leq 1$ by $D_\gamma$ we denote the corresponding LSA (cf. [1]).

Recall that in [2] it is proved that one can obtain $\beta$-PLSA recognizing the same language $L$ as $D$ for any constant $\beta > 0$ increasing the runtime of $D$ by at most a constant factor. We shall use this remark to make $\alpha$ as small as desired.

As in [1] one shows that for any vertex of $L$ there exists $\epsilon > 0$ such that each hyperplane occurring as a testing one in $D$ which intersects the closed ball $B_\epsilon(v)$ of the radius $\epsilon$ and with the center in $v$, should pass through $v$. 

Similar to [1] select from $D$ all the testing hyperplanes passing through $v$. Then the obtained thereby $D'$ is an $-PLSA$ recognizing the language $L \cap B_r(v)$, when being restricted on $B_r(v)$.

Making a suitable affine transformation, we can assume that $v$ is the coordinate origin and besides, the hyperplanes from $L$ passing through $v$ are just the coordinate hyperplanes $\{X_1 = 0\}, \ldots, \{X_n = 0\}$.

For any $0 \leq \gamma \leq 1$ each leaf of $D_\gamma'$ provides a polyhedra $V$ of the form
\[ \{L_1 = 0\} \cap \ldots \cap \{L_{q_1} = 0\} \cap \{L_{q_1+1} > 0\} \cap \ldots \cap \{L_q > 0\} \]
for some testing hyperplanes $L_1, \ldots, L_q$. Then $P = \{L_1 = \ldots = L_q = 0\}$ is the minimal (in the sense of inclusion) face of the closure of $V$. If $q_1 = 0$ then $V$ is open. Polyhedra corresponding to all the leaves of $D_\gamma'$ form the partition $\mathbb{R}^\omega$.

For the time being we fix $0 \leq \gamma \leq 1$ and an open polyhedron $V$. Denote by $\Delta(V)$ the maximal dimension of the faces of $L$ passing through $v$ which are contained in $P$. Any such face of $L$ has the form $\bigcap_{i \in I} \{X_i = 0\}$ for a certain subset $I \subseteq \{1, \ldots, n\}$. Observe that if two faces $\bigcap_{i \in I} \{X_i = 0\}$ and $\bigcap_{i \in I \cap J} \{X_i = 0\}$ of $L$ are contained in $P$ then the face $\bigcap_{i \in I \cap J \cap K} \{X_i = 0\}$ is contained in $P$ as well. Thus, there is the unique maximal face of the form $\bigcap_{i \in I} \{X_i = 0\}$ contained in $P$ and its dimension equals to $\Delta(V)$.

## 2 Estimating spherical measure of intersections of a polyhedron with the coordinate hyperplanes

For any set $W \subseteq \mathbb{R}^\omega$ consider its cone $C(W)$ with the vertex in the origin and by $\delta_n(W) = \mu_n(C(W) \cap B_1)/\mu_n(B_1)$ where $\mu_n$ is the usual Borel measure in $\mathbb{R}^\omega$ and the ball $B_1 = B_1(0)$ (we consider only measurable sets).

Take any line $h \in P$ passing through the origin (provided that $\dim P > 0$ and such a line does exist) and let $H$ be a hyperplane orthogonal to $h$ and passing through the origin.

**Lemma 1.** $\delta_n(V) = \delta_{n-1}(V \cap H)$

**Proof.** Actually, a more general statement holds. For any subset $U \subseteq H$ for the direct product $U \times h \subseteq \mathbb{R}^\omega$ we have $\delta_n(U \times h) = \delta_{n-1}(U)$. To prove the latter statement one can consider a partition of $H \cap B_1 = U \cup U \ldots \cup U_t$ into "small" pieces where $U_i = R_i(U_0)$, $1 \leq i \leq t$ for appropriate rotations $R_i$ of $H$. Extend every $R_i$ to the rotation $R_i$ of $\mathbb{R}^\omega$ by leaving $h$ invariant. Then $1 = \delta_n(B_1) = (t+1)\delta_n(U_0 \times h)$ and $1 = \delta_{n-1}(H \cap B_1) = (t+1)\delta_{n-1}(U_0)$. The standard arguing with approximation of $U$ by a partitioning into "small" pieces completes the proof of the lemma. 

\[ \square \]
Lemma 2. If $\alpha$-PLSA $D^l$ recognizes the language $L \cap B_r(v)$ (where $L$ is in a generic position), being restricted on $B_r(v)$, where $v$ is a vertex of $L$, then with a probability $\geq p = 1 - \frac{2n}{cn}$ (thus, we assume that $\alpha < \frac{c^2}{2}$, see the remark in section 1), a certain leaf of $D^l_V$ provides an open polyhedron $V$ with $\Delta(V) \leq c_3 n$.

Proof. Suppose the contrary. Recall that we assume that $v$ coincides with the origin and among the hyperplanes $H_1, \ldots, H_m$ there are $\{X_1 = 0\}, \ldots, \{X_n = 0\}$. Then $1 = \sum \delta_n(V)$ where the summation ranges over all open polyhedra $V$ provided by the leaves of $D^l_V$. Assume that for a particular value of the random parameter $0 \leq \gamma \leq 1$ for all open $V$ we have $\Delta(V) > c_3 n$.

Let $P$ be the minimal face of $V$, then $P \subset \{X_{i_1} = \ldots = X_{i_l} = 0\}$ for some indices $1 \leq i_1, \ldots, i_l \leq n$ with $l < (1 - \alpha) n$. For any index $j \notin \{i_1, \ldots, i_l\}$ lemma 1 entails $\delta_{n-1}(V \cap \{X_j = 0\}) = \delta_{n}(V)$. Therefore $\sum_{V} \sum_{1 \leq l \leq n} \delta_{n-1}(V \cap \{X_l = 0\}) > c_3 n$. By the supposition the expectation of the latter sum over the values of the random parameter $0 \leq \gamma \leq 1$ is greater than

$$E \left( \sum_{V} \sum_{1 \leq l \leq n} \delta_{n-1}(V \cap \{X_l = 0\}) \right) > (1 - p)c_3 n = 2\alpha n.$$ 

This contradicts to the definition of $\alpha$-PLSA taking into account that for any point from $V \cap \{X_l = 0\}$ the output of $D^l_V$ is the same as for the points, from its small neighbourhood, so $D^l_V$ does not distinguish them. The obtained contradiction proves the lemma.

3 Lower bound on the number of faces in PLSA

Now we complete the proof of the theorem, the arguing is similar to one in [1]. Applying lemma 2 to each vertex of $L$ we conclude that there exists a value $0 \leq \gamma \leq 1$ of the random parameters such that for at least $p \gamma$ vertices $v$ of $L$ there is an open polyhedron $V$ provided by corresponding to a leaf of $D^l_V$ such that $V$ has a face $P$ (which could be not a minimal face of $P$ unlike the local situation in section 2) containing $v$ and if some $l$-face of $L$ is contained in $P$ and contains $v$ then $l \leq c_3 n$. To every such vertex $v$ let us correspond a face $p$ (if there are several such faces then correspond any of them).

Since $L$ is in a generic position (see the definition), any face $P$ of $D^l_V$ could be corresponded to at most $m^c n^2$ vertices of $L$. Hence there are at least $p \gamma/m^c n^2 = pm(n^{c_1} - c_2 n)$ faces of $D^l_V$. But on the other hand, the number of faces in $D^l_V$ does not exceed $2^{2T}$ (cf. [1]), therefore $2^{2T} \geq pm(n^{c_1} - c_2 n)$, this completes the proof of the theorem.
References
