o-Minimal Expansions of the Real Field: A Characterization, and an Application to Pfaffian Closure

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Abstract

Using a modification of Wilkie's recent proof of o-minimality for Pfaffian functions, we gave an invariant characterization of o-minimal expansions of $\mathbb{R}$. We apply this to construct the Pfaffian closure of an arbitrary o-minimal expansion of $\mathbb{R}$.

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0 Introduction

While working on complexity questions for VC-dimension of general neural networks and corresponding semi-Pfaffian sets [KM97a], we became convinced that major progress on o-minimality should come from a more systematic use of Sard's Theorem and Morse Theory [H76]. The importance of these for Khovanski's basic work [K91] is clear. Our result was strongly confirmed by Wilkie's 1996 result [W96] on the o-minimality of expansions of the real field by C^∞ primitives, provided that the (necessary) condition holds that all quantifier-free definable sets have finitely many connected components, uniformly in parameters. A striking consequence is the o-minimality of the Pfaffian C^∞-structure on $\mathbb{R}$.

In this paper we go further, in particular removing, exactly to the extent to which this is possible, Wilkie's C^∞ assumption. Wilkie used Sard, but not Morse. We use both, and an important recent result of van den Dries and Miller [DM96] (which abstracts an important result of Bierstone, Milman and Pawlucki).

Our original goal was merely to relativize Wilkie's result, by showing that if M is an o-minimal expansion of the real field then the expansion of M by the addition of functions "Pfaffian over M" (discussion and exact definition in section 5) is o-minimal.

For M based on C^∞ primitives this is quite easy, using Wilkie, plus Sard and Morse in Khovanski's style. To do it without the C^∞ assumption needed a new idea, coming from [DM96]. Then, having achieved this objective, we realized that we had obtained a nice characterization of o-minimal expansions of the real field.

In this paper we reverse the order of discovery. The characterization is Theorem 4 below. The application to Pfaffian closures is Theorem 22.

1 Background on o-minimality

1.1 Following an old (and recently revived) tradition, we avoid syntactical considerations while doing pure model theory (especially definability theory). Of course if we move on to complexity considerations syntax is relevant.

1.2 For X a nonempty set and n an integer ≥ 1, X^n is the usual cartesian power. For $n = 0$, it is natural to put $X^0 = \{\emptyset\} = 1$. An n-ary function on X is identified with a subset of X^{n+1} (satisfying functionality conditions). So a 0-ary function is naturally associated to a constant.

1.3 If σ is a map $n \to m$, we write σ^* for the map $X^m \to X^n$ given by $(x_1, \ldots, x_m) \to (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

If $n \leq m$, and σ is the inclusion, σ^* is the usual projection $\pi_{m,n} : X^m \to X^n$.

1.4 As in [W96] our basic objects are X-systems, i.e. sequences $\langle S_n \rangle_{n \in \omega}$, where S_n is a set of subsets of X^n. A is in S if A is in some S_n.

If S is an X-system, X is *pure* if it satisfies the axioms:

(P1) $X \in S$;

(P2) Each S_n is closed under σ^*, where σ is a permutation of n;

(P3) $A \in S_n \to A \times X \in S_{n+1}$;

(P4) S_n closed under $\cap$.

It then follows that if $A \in S_n$ and $B \in S_m$ then $A \times B \in S_{n+m}$ [W96].

Wilkie defines two operations on X-systems, and shows that the class of pure X-systems is closed under both. These are:

- $S^{\cup} = \langle S_n^{\cup} \rangle$, where $S_n^{\cup}$ is the closure of S_n under finite unions;
- $S^{\exists} = \langle S_n^{\exists} \rangle$, where $S_n^{\exists}$ is the collection of all $\pi_{m,n}[A]$, $m \geq n$, $A \in S_m$.

Obviously $S_n \subseteq S_n^{\cup}$, and $S_n \subseteq S_n^{\exists}$.

Write $\subseteq$ for the obvious (pointwise defined) partial order on X-systems. Let $Pure(S)$ be the least pure system extending S (it obviously exists).

1.5 Now assume X has a fixed topology, inducing the product topologies on X^n. Let $\overline{}$ denote closure, in appropriate contexts.

Here we modify Wilkie's notation slightly. Let S_n^{LC} consist of all $A_1 \cap \overline{A_2} \cap \cdots \cap \overline{A_r}$, $A_i \in S_n$, and $S^{LC} = \langle S_n^{LC} \rangle$ (LC stands for *local closure*). Then Wilkie showed that $S \subseteq S^{LC}$, and LC preserves purity [W96].

1.6 Finally we go to $X = \mathbb{R}$, with the order topology, and to what Wilkie calls *tame* systems. There are four axioms **(T1)** – **(T4)**, of which only the last requires discussion. Firstly:

(T1) S is pure;

(T2) S_n contains every semi-algebraic subset of $\mathbb{R}^n$ (and in particular every singleton);

(T3) For every n and $A \in S_n$ there exists $m \geq n$ and a closed $B \in S_m$ with $A = \pi_{m,n}[B]$.

(T4) has to do with the type of finiteness condition found by Khovanski [K91], which was so inspirational for Wilkie's breakthrough in [W94].

Suppose $A \subseteq \mathbb{R}^{k+l}$, and $\tilde{\beta} \in \mathbb{R}^l$. Let $A_{\tilde{\beta}} = \{\overline{x} : (\overline{x}, \tilde{\beta}) \in A\} \subseteq \mathbb{R}^k$. Give $A_{\tilde{\beta}}$ the subspace topology, and let $b_o(\tilde{\beta})$ = number of connected components of $A_{\tilde{\beta}}$. Following [K91] we are interested in situations where $b_o(\tilde{\beta})$ is bounded independently of $\tilde{\beta}$.

Of course if only $k + l$ (not k, l) is given, we would wish to consider all permutations σ of $k + l$, and all k', l' with $k' + l' = k + l$. This would give only finitely many variations on the original case, and one would want a b_o bound covering all cases simultaneously. Since we have **(T1)**, **(T2)** anyway, it is then reasonable to impose:

(T4) If $A \in S^n$, there is an integer N such that for any affine subset Y of R^n, $\overline{A} \cap Y$ has $\leq N$ connected components in the subspace topology.

[Recall that an affine function on $\mathbb{R}^n$ is the sum of a linear function and a constant, and that an affine subset is the zeroset of any set of affine functions. But obviously this is the same notion as zeroset of $\leq n$ affine functions].

Khovanski proved that **(T4)** holds for any semi-Pfaffian set [K91], and **(T1)** – **(T3)** are trivial in this situation.

The point of Wilkie's formulation of **(T4)** rather than the version discussed earlier is that it is useful in inductive arguments (where he makes ingenious use of it).

1.7 Wilkie shows that the class of tame $\mathbb{R}$-systems is closed under $\cup, \exists, LC$ [W95].

Let S be a tame $\mathbb{R}$-system. Then there is a smallest $\mathbb{R}$-system extending S and closed under $\cup, \exists, LC$. This system is tame. We propose to call it $Ch(S)$, the Charbonel closure of S, in recognition of the insights in [C91].

1.8 $Ch(S)$ has some agreeable topological and measure-theoretic properties. Let μ be Lebesque measure, and μ_n the corresponding product measure $\mathbb{R}^n$. Then, easily, every set in $(Ch(S))_n$ is μ_n-measurable. Moreover, any set in $(Ch(S))_n$ with no interior has measure 0. And, finally, if a set in $(Ch(S))_n$ has no interior its closure has measure 0, and in particular no interior. See [W95].

1.9 **Tarski systems** After these rather nontrivial considerations, we return temporarily to generalities. X will be again a nonempty set.

Definition: S is a Tarski system on X if S is a pure X-system such that $S = S^{\cup} = S^{\exists}$, and such that each S_n is a Boolean subalgebra of X^n, and the graph of $=$ is in S_2.

A Tarski system on X is closed under the usual primitives of first order logic, and is in fact no more than a sequence $\langle S_n \rangle$ such that for some language L and L-structure on X, S_n is the collection of definable subsets of X^n [DM96].

Note that complement (or negation) was not involved in the discussion through §1.7 . The extraordinary fact is that under some mild assumptions on tame $\mathbb{R}$-systems S, $Ch(S)$ is a Tarski system [W96].

1.10 ***o*-minimality on $\mathbb{R}$** Let S be a Tarski system on $\mathbb{R}$, with the graph of $>$ in S_2.

Definition: S is o-minimal if every $A \in S$, has finite boundary.

Most remarkably, one knows:

Theorem: [D97] If S is o-minimal then for every $A \in S_{k+l}$, $b_0(\tilde{\beta})$ is bounded uniformly for $\tilde{\beta} \in \mathbb{R}^l$ (cf. §1.5).

If then S contains the semilinear sets, **(T4)** holds. Also, the connected components of $A_{\bar{\beta}}$ are in S, when S is o-minimal. For a beautiful survey of the many important consequences of o-minimality, see [DM96].

1.11 It is of course trivial that tame Tarski systems on $\mathbb{R}$ are o-minimal. But one now knows deep results that go from tame S to o-minimal $Ch(S)$, and we now turn to them.

Recall our convention that functions are (unless otherwise specified) identified with relations.

If S is an X-system, there is obviously a smallest Tarski system on X extending S. We call it $Tarski(S)$ (the Tarski closure of S). A Tarski system S' is generated by a system S if $S' = Tarski(S)$.

Note that if a Tarski system S' is generated by S, then S' consists of the definable relations on the natural L-structure on X where L has $=$, function symbols for the members of S which are total functions, and relation symbols for the other relations in S.

Definition: ($0 \leq j \leq \infty$) An $\mathbb{R}$-system S is C^j if its members are either $>$ or C^j total functions.

Finally:

Theorem 1: [W96]. Let S be a C^∞ $\mathbb{R}$-system, such that $Pure(S)$ is tame. Then $Ch(S) = Tarski(S)$ and is o-minimal.

We now set out to improve this, by weakening the C^∞ assumption.

2 Generating o-minimal $\mathbb{R}$-systems

2.1 We do not know if every o-minimal $\mathbb{R}$-system is of the form $Tarski(S)$, for S C^∞. This seems to us unlikely. We are however inclined to conjecture that every o-minimal S' extends to one of the form $Tarski(S)$, for S C^∞. If this were proved, it would simplify greatly our work in section 5.

2.2 It is of course trivial that any o-minimal S' is of the form $Tarski(S)$, where S consists of order and total functions (just use characteristic functions). This is uninteresting, and useless. However:

Theorem 2: If S' is o-minimal, then for $0 \leq j < \infty$ there is an S such that $S' = Tarski(S)$ and S is C^j.

Proof: Here we start using substantial facts about o-minimality.

Fix j, and let S consist of $>$ and all total C^j functions in S'. We claim $S' = Tarski(S)$.

By [D97], every A in S' is a finite union of cells. So we have only to show that cells are in $Tarski(S)$. This is now done by induction, following the inductive definition of cell in [D97].

For points, or cells on the line, the result is obvious, as these need only $=$ and $>$ in their definition. At a later stage of the induction, to define a cell A in $\mathbb{R}^n$ one has a cell B in $\mathbb{R}^{n-1}$ (which we can assume to be on $Tarski(S)$), and either

1. an element f of S' which is a continous function $B \rightarrow \mathbb{R}$, and $A = f$,
2. elements of f, g of S', both continous functions $B \rightarrow \mathbb{R}$, with $f < g$ on B, and A is

$$\{(x_1, \ldots, x_{n-1}, x_n) : (x_1, \ldots, x_{n-1}) \in B \\ \wedge f(x_1, \ldots, x_n) < x_n < g(x_1, \ldots, x_n)\}$$

We have to show A is in $Tarski(S)$.

In Case 1, consider the closed set $\overline{A}$. By a crucial result of [DM96], $\overline{A}$ is the zero set of a total C^j function in S', and so $\overline{A}$ is in $Tarski(S)$. But $A = \overline{A} \cap (B \times \mathbb{R})$, so is in $Tarski(S)$.

For Case 2, get f, g in $Tarski(S)$ by above argument. Clearly A is definable from these and $<$.

□

2.3 The Zero Set Condition. We have just seen that any o-minimal S' is generated, for any $j \geq 0$, by $>$ and a set of total C^j functions. We made crucial use of the following result [DM96] (inspired by unpublished work of Bierstone, Milman and Pawlucki):

Theorem 3: If S' is o-minimal, and A is a closed set in S, then for each $j \geq 0$, A is the zeroset of some total C^j f in S'.

This now suggests our main definition. Before it:

Definition: If f is a function $\mathbb{R}^n \to \mathbb{R}$, $Zer(f)$ is the zeroset of f, and $Pos(f)$ the positivity set of f.

Finally:

Definition: Let S be an $\mathbb{R}$-system of total continous functions. S is an o-minimal generator if:

(1) S containing the constant functions, and each π_m, and is closed under $+, -, \cdot$.

(2) If $f \in S_{m+1}$ (i.e. $f : \mathbb{R}^m \to \mathbb{R}$) and σ is a map $m \to n$, then $\sigma^* f \in S_{n+1}$;

(3) (The Zero Set Condition) If f is in S then for each $j \geq 1$ there is g in S such that g is C^j and $Zer(f) = Zer(g)$;

(4) (Uniform H_0 bounds) Let $n = k + l$, and $f_1, \ldots, f_r, g_1, \ldots, g_S \in S_{n+1}$. Let $A = Zer(f_1) \cap \ldots \cap Zer(f_r) \cap Pos(g_1) \cap \ldots \cap Pos(g_s)$. Then as $\tilde{\beta}$ varies in $\mathbb{R}^l$, $b_0(\tilde{\beta})$ is bounded.

(5) If f is in S, f C^1, then its partial derivatives are in S.

Our main theorem is:

Theorem 4: Let S' be a Tarski system with $>$ in $(S')_2$, and contains all constants. Then S' is o-minimal if and only if for some o-minimal generator S, $S' = Tarski((S, >))$, where $(S, >)$ is the system generated by S and $>$.

We already know necessity, by Theorem 2 and the basic uniformity of connected components for o-minimal theories.

In the next section we prove Theorem 4 (and more). Sard's Theorem will now come into play.

3 Sardian Considerations

3.1 Let us fix an o-minimal generator S. Let T be the system such that T_n consists of all sets A as in clause **(4)** above.

Lemma 5: T is pure.

Proof:

(P1) $\mathbb{R} = Zer(0)$.

(P2) By clause **(2)**.

(P3) By clause **(2)**.

(P4) Clear.

□

But even more:

Lemma 6: $T^{\cup}$ is tame.

Proof: **(T1)** and **(T2)** are clear. **(T3)** is essentially the observation

$$y > 0 \Leftrightarrow (\exists u, v)(y - v^2 = 0 \wedge u \cdot v - 1 = 0)$$

and is an old idea going back to [T51].

(T4) is an easy consequence of Clause 4, the presence of linear functions, and that the union of two sets each with finitely many connected components has finitely many connected components.

□

Corollary: $Ch(T^{\cup})$ is tame.

Proof: Theorem 1. □

3.2 Now our task is to show that $Ch(T^{\cup}) = Tarski(T^{\cup})$, and is *o*-minimal. As Wilkie shows, it suffices to show:

> (*) For any closed A in $Ch(T^{\cup})$, there is a closed B in $Ch(T^{\cup})$ such that B has empty interior and $\partial A \subseteq B$.

The passage from (*) to our conclusion (closure under complement for $Ch(T^{\cup})$) does not need any differentiability assumptions. But the proof of (the analogue of) (*) in Wilkie [W96] uses a C^{∞} assumption. We will get by with the ZeroSet Condition.

We get from T to $Ch(T^{\cup})$ by iteration of $\cup, \exists$ and LC, and (*) will be proved via an induction on how A is generated. In fact something much more precise has to be proved by this type of induction.

3.3 Tameness and Sard. Wilkie discovered that inside tame systems Sard's Theorem can be freed from the usual restriction (that if $f : \mathbb{R}^m \to \mathbb{R}^n$ and f is C^k where $k > \max(0, m-n)$ then the set of nonregular values has measure 0). In fact:

Theorem 7: Suppose S is a tame $\mathbb{R}$-system. Suppose $n \geq m \geq 1$, and $F : U \to \mathbb{R}^m$ is a C^1 function, where U is open in $\mathbb{R}^n$ and U and F are in S. Then the set of nonregular values of F is in $Ch(S)$, and has measure 0.

Proof: See [W96] and §1.7.

□

And similarly:

Theorem 8: Suppose S is a tame $\mathbb{R}$-System. Let $n > m \geq 1$, $\bar{a} \in \mathbb{R}^m$. Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}$ are in S and C^1. Then if $\bar{a}$ is a regular value for F there are at most finitely many $b \in \mathbb{R}$ such that $(\bar{a}, b)$ is a nonregular value for $\langle F, f \rangle$.

3.4 Maxwell's thesis. As a contribution to Wilkie's programme his student, Steve Maxwell [M96] showed the following for tame $\mathbb{R}$-systems:

Theorem 9: (Weak selection) (S tame)
Suppose $n, m \geq 1$, $A \in S_n$, $B \in S_{n+m}$, and A has nonempty interior. Suppose $\forall \bar{x} \in A \exists \tilde{y} \in \mathbb{R}^m\ \langle \bar{x}, \tilde{y} \rangle \in B$. Then there is a nonempty open set $U \subseteq A$, U in $(Ch(S))_n$, and a $\varphi : U \to \mathbb{R}^m$, φ in $Ch(S)$, such that $\forall \bar{x} \in U\ \langle \bar{x}, \varphi(\bar{x}) \rangle \in B$.

Theorem 10: (Almost everywhere smoothness) (S tame)
Suppose $n, m \geq 1$, $N \geq 0$, U open, $U \in S_n$, and $F : U \to \mathbb{R}^m$, F in S. Then there is a closed set A in $Ch(S)_n$, containing no interior points, such that F is C^N on $U \setminus A$.

Theorem 11: (Closure under differentiation) (S tame)
Suppose $n \geq 1$, $U \in S_n$, U open, and $F : U \to \mathbb{R}$ a C^1 function in S. Then the partial derivatives (on U) are in $Ch(S)$.

3.5 The last needed result about tame S is

Theorem 12: (S tame)
Suppose that $n, k \geq 1$ and $F : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is a C^1 function in S. Suppose $\bar{a}$

is a regular value of F, and U is an open ball in $\mathbb{R}^n$ with the property that $X = F^{-1}(\bar{a}) \cap (U \times \mathbb{R}^k)$ is nonempty and bounded. Then either

(i) $\pi[X] = U$, where $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ is the projection map onto the first n coordinates,

or

(ii) there exists $\eta > 0$ and distinct $1 \leq i_1 < i_2 < \ldots < i_k \leq n+k$ such that

$$\det\left(\frac{\partial(F_1, \ldots, F_k)}{\partial(x_{i_1}, \ldots, x_{i_k})}\right) \restriction X$$

takes all values in $[0, \eta]$ (here $F = (F_1, \ldots, F_k)$).

Proof: [W96]

□

4 Approximation Theory

4.1 We now follow Wilkie in setting up definitions and a goal for approximation theory, but we will prove a different result.

4.2 Definition: (as in [W96])

(i) $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$;

(ii) ($k \in \mathbb{N}$) A k-modulus is a sequence $\overline{\mu} = \langle \mu_0, \ldots, \mu_k \rangle$ with $\mu_0 \in \mathbb{R}_+$, and $\mu_i : \mathbb{R}_+^i \to \mathbb{R}_+$ for $i = 1, \ldots, k$;

(iii) If $\overline{\mu}$ is a k-modulus, and $\overline{\epsilon} = \langle \epsilon_0, \ldots, \epsilon_k \rangle \in \mathbb{R}_+^{k+1}$, then $\overline{\epsilon}$ is $\overline{\mu}$-bounded if $\epsilon_0 < \mu_0$ and $\epsilon_i < \mu_i(\epsilon_0, \ldots, \epsilon_{i-1})$ for $i = 1, \ldots, k$;

(iv) If $\overline{x} = \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$, $\|\overline{x}\| = \max\{|x_i| : 1 \leq i \leq n\}$;

(v) Suppose $n \geq 1$, $k \in \mathbb{N}$, $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^{n+k}$, and $\overline{\mu}$ is a k-modulus. Then $B \leq A(\text{mod } \overline{\mu})$ (B approximates A from below (mod $\overline{\mu}$)) if for all $\overline{\mu}$-bounded $\overline{\epsilon}$ ($= \langle \epsilon_0, \ldots, \epsilon_k \rangle$) and all $\tilde{x} \in \mathbb{R}^n$, if $\langle \tilde{x}, \epsilon_1, \ldots, \epsilon_k \rangle \in B$ then there is $\tilde{y}$ in A with $\|\tilde{x} - \tilde{y}\| < \epsilon_0$;

(vi) (Same assumptions as in **(v)**).
$A \leq B(\text{mod } \overline{\mu})$ (B approximates A from above on bounded sets (mod $\overline{\mu}$)) if for every $\overline{\mu}$-bounded $\overline{\epsilon}$ $(= \langle \epsilon_0, \ldots, \epsilon_k \rangle)$ and every $\tilde{x}$ in A with $\|\tilde{x}\| < \epsilon_0^{-1}$ there is $\tilde{y}$ in $\mathbb{R}^n$ with $\|\tilde{x} - \tilde{y}\| < \epsilon_0$ and $\langle \tilde{y}, \epsilon_1, \ldots, \epsilon_k \rangle \in B$.

4.3 The main use of the above notions comes from

Lemma 13: [W96]. (S tame). Suppose $n \geq 1$, $k \in \mathbb{N}$, $A \in S_n$, $B \in S_{n+k}$ and B has empty interior. Let $\overline{\mu}$ be a k-modulus such that $\partial A \leq B(\text{ mod } \overline{\mu})$. Then there is in $(Ch(S))_n$ a closed set C with empty interior such that $\partial A \subseteq C$.

Proof: [W96]

□

The problem is how, given A, and seeking C, one constructs B (evidently, by [W95], one needs only that B is in $Ch(S)$). Wilkie uses a C^∞ assumption. We are going to use the assumption that $S = T^\cup$, for T an o-minimal generator.

In both approaches, a crucial role is played by Sardian considerations.

4.4 Sardian sets over $\mathcal{S}$ Let $\mathcal{S}$ be an o-minimal generator.

Definition: Let $B \subseteq \mathbb{R}^n \times \mathbb{R}^k_+$, with $n, k \geq 1$. B is l-Sardian over $\mathcal{S}$ if B is defined by

$$(x_1, \ldots, x_n, \varepsilon_1, \ldots, \varepsilon_k) \in B \Leftrightarrow (\exists x_{n+1}) \cdots (\exists x_{n+k-1}) \bigwedge_{i=1}^{k} f_i(x_1, \ldots, x_{n+k-1}) = \varepsilon_i$$

where the f_i are $C^l : \mathbb{R}^{n+k-1} \to \mathbb{R}$, and are in $\mathcal{S}$.

Remarks:

(i) In the above, let $f = (f_1, \ldots, f_k)$. Suppose also $l \geq 1$. Then the set of nonregular values $\overline{\varepsilon}$ for f has measure zero, and is in $Ch(T^\cup)$. This is because of [W96];

(ii) In the above, B has no interior [W96];

(iii) In the above, B is in $Ch(T^\cup)$, obviously.

4.5 The main result

Theorem 14: ($\mathcal{S}$ an o-minimal generator).
For A in $Ch(\mathcal{S})$, and all $l \geq 1$, there exists a $k \geq 1$ (an l-complexity for A), a k-modulus $\overline{\mu}$ (an l-modulus for A), and a set $B \subseteq \mathbb{R}^n \times \mathbb{R}_+^k$ (an l-approximation for A) such that B is a finite union of l-Sardian sets over $\mathcal{S}$ (l-approximating constituents for A) such that $\partial\overline{A} \leq B(\text{mod } \overline{\mu})$ and $B \leq \partial\overline{A}(\text{mod } \overline{\mu})$.

Once we have proved this, there will follow, exactly as in [W96],

Corollary: ($\mathcal{S}$ an o-minimal generator).
$Ch(\mathcal{S}) = Tarski(\mathcal{S})$, and is o-minimal.

The theorem is proved by induction essentially on the construction of A in $Ch(\mathcal{S})$.

In the previous discussion we have worked over $T^{\cup}$, since it is tame. We will take this as the base of our constructions.

We will prove Theorem 14 by induction on how A is constructed. Typically we will have the form of the result for all A and all l, and A in some tame system T' between $T^{\cup}$ and $Ch(T^{\cup})$. Then we have to go on to $(T')^{\cup}$, or $(T')^{\exists}$, or $(T')^{LC}$. Typically, to get the l result for an entity in the new system, we will use the $(l+1)$-result on entities in the earlier system. In fact, the path is not quite as straight as this outline suggests.

We now list the essential modules involved.

(1) If $A_1, \ldots, A_r$ have the property for l, so does $A_1 \cup \cdots \cup A_r$.

This is obvious, and something similar is done in [W96].

(2) If $A \subseteq \mathbb{R}^{n+1}$, and has the $(l+1)$-property, then $\pi[A]$ has the l-property, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates.

This is (essentially) done in 3.10 of [W96], and uses the fact that $\mathcal{S}$ contains

$$\left(1 + \sum_{i=n+1}^{n+k} x_i^2\right)^{\frac{1}{2}}$$

all n, k, and is an algebra closed under any partials which exist.

We stress that this module requires work, done in [W96].

(3) Suppose f in $\mathcal{S}$, and $A = Zer(f)$. Then A has the l-property for all l.

Here we use critically the ZeroSet Condition, to get $A = Zer(g)$, g in $\mathcal{S}$, g C^l. Then use also that

$$\left(\left(1+\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} x_{n+1}^2\right)^{-1}$$

is in $\mathcal{S}$, all n. Then the argument in 3.8 of [W96] works.

(4) If A is in T, A has the l-property, all l.

The basic idea is that for f in S, $Pos(f)$ is the projection of some $Zer(g)$, g in S. Thus A in T is also of this form. Now use modules **(2)** and **(3)**.

(5) If A is in $T^{\cup}$, A has the l-property, all l.

Use modules **(4)** and **(1)**.

(6) Now we use a cunning idea of Wilkie [W96].

Suppose $A \subseteq \mathbb{R}^n$ has the l-property. Let Y be an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$, such that $\overline{A} \cap Y = \partial(\overline{A}) \cap Y$. Then $\overline{A} \cap Y$ has the l-property.

This is done in 3.12 of [W96]

(7) The reader would be right to regard $T' \mapsto (T')^{LC}$ as the most problematic step.

However, Wilkie, in 1.19 of [W96], remarked that for tame systems T', and A in $(T')^{LC}_n$, there exists $m \geq n$ and A' in T'_m, and an affine subspace Y of $\mathbb{R}^m$ such that $A = \pi[\overline{A'} \cap Y]$, where π is the projection onto the first n coordinates.

So, because of module **(2)**, we need, to handle LC, only the following:

Suppose $A \subseteq \mathbb{R}^n$ has the l-property, and Y is an affine subspace of $\mathbb{R}^n$. Then $\overline{A} \cap Y$ has the l-property.

Wilkie does this, using module **(6)**, in 3.13.

These are our modules, and it follows easily that our Theorem 14 holds.

□

4.6 A remark about model completeness. Let S be an o-minimal generator, and L the language based on S and $>$. Then:

Lemma 15: Every A in $Ch((S,<))$ is existentially definable (for L), provided the closure of an existentially definable set is existentially definable.

Proof: Obvious. □

Corollary: $Tarski\,((S,<))$ is model complete if and only if the closure of any existentially definable set is existentially definable.

5 Pfaffian Closure

5.1 Let us first recall Khovanski's definition of Pfaffian chain, and Pfaffian function [K91].

Definition: Suppose $f_0, \ldots, f_{k-1}$ are real analytic functions on $\mathbb{R}^n$. $\langle f_0, \ldots, f_{k-1} \rangle$ is a Pfaffian chain of length k if for $i < k$ and $i \leq j \leq n$ there are polynomials $P_{ij}\,(x_1, \ldots, x_n, y_1, \ldots, y_i)$. such that for each i, j $\frac{\partial f_i}{\partial x_j} = P_{ij}\,(x_1, \ldots, x_n, f_0(\overline{x}), \ldots, f_i(\overline{x}))$.

Definition: f is Pfaffian if it occurs in a Pfaffian chain.

Examples are polynomials, e^x, arctan, $\int_0^x e^{-t^2} dt$, $(1 + \sum_{j=1}^n x_j^2)^{\frac{1}{2}})$, and $\left((1 + \sum_{j=1}^n x_j^2)^{\frac{1}{2}}) + x_n^2\right)^{-1}$.

5.2 It is quite easy to see that the collection of Pfaffian functions is closed under $+$, $\cdot$, $-$.

Of fundamental importance is Khovanski's Theorem [K91] which can be (re)written as:

The class of Pfaffian functions has the property of uniform H_0-bounds.

It follows easily that the $\mathbb{R}$-system generated from all $Zer(f)$ and $Pos(f)$, f Pfaffian, using $\wedge$ and $\vee$, is tame. In this way, using his new techniques, Wilkie proved:

Theorem 16: The Tarski system generated by $>$ and the Pfaffian functions is o-minimal.

5.3 Many people have noticed, in varying generality, that Khovanski's argument relativizes to cases where the polynomials are replaced by some other class of functions having the property of uniform H_0-bounds. However, as far as we know, the treatment below is the most general available.

First we make a crude relativization.

Definition: (S an $\mathbb{R}$-system of functions).

i) Let $f_0, \ldots, f_{k-1}$ be differentiable functions on $\mathbb{R}^n$. $\langle f_0, \ldots, f_{k-1} \rangle$ is a Pfaffian chain over S of length k if for each $i < k$ and $i \leq j \leq n$ there are $P_{ij}(x_1, \ldots, x_n, y_1, \ldots, y_i)$ in S such that for each i, j $\frac{\partial f_i}{\partial x_j} = P_{ij}(x_1, \ldots, x_n, f_0(\overline{x}), \ldots, f_i(\overline{x}))$.

ii) f if Pfaffian over S if f occurs in a Pfaffian chain over S.
Note: The f_i are not assumed C^∞.

At this level of generality one will not be able to transfer finiteness properties from S to the class of Pfaffians over S.

Let us make some obviously reasonable restrictions on S, namely that it consists of continuous functions, contains all constant functions, all σ^*, and that each S_n is a ring under $+, \cdot, -$.

Suppose $Q_1, \ldots, Q_n$ are in S, functions from $\mathbb{R}^{n+k}$ to $\mathbb{R}$, all C^1. Let $\langle f_0, \ldots, f_{k-1} \rangle$ be a Pfaffian chain over S, with the P_{ij} as above, and we assume the F_i are C^1. Then, following Khovanski, one seeks to bound (with appropriate uniformities) the number of regular points for

$$\langle Q_1(\overline{x}, f_0(\overline{x}), \ldots, f_{k-1}(\overline{x})), \ldots, Q_n(\overline{x}, f_0(\overline{x}), \ldots, f_{k-1}(\overline{x})) \rangle$$

in the inverse image of $\overline{O}$ ($\in \mathbb{R}^n$). Again following [K91] one considers the map $(F_1, \ldots, F_n, G)$ $(= (F, G))$ from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$, where $F_i(x_1, \ldots, x_n, x_{n+1}) = Q_i(x_1, \ldots, x_n, f_0(\overline{x}), \ldots, f_{k-2}(\overline{x}), x_{n+1})$ and $G(x_1, \ldots, x_n, x_{n+1}) = f_{k-1}(\overline{x}) - x_{n+1}$. So (F, G) is again a C^1 map. Let M be the C^1-submanifold of $\mathbb{R}^{n+1}$ given by $G = 0$ (M is the graph of f_{k-1}).

The Jacobian J of (F, G) is continous and in S. $\hat{J}$, its restriction to M, is the restriction to M of some $Q(\overline{x}, f_0(\overline{x}), \ldots, f_{k-1}(\overline{x}), x_{n+1})$, where Q is a continous element of S. (To get more differentiability on Q, we need more on the Q_i and on the P_{ij}).

By adding if need be a polynomial condition, with a parameter, we can assume (without changing the chain) that the original map, and (F, G), are proper. n is changed to $n+1$, thereby affecting bounds, but this is irrelevant to the present informal analysis.

Let $\tilde{F}$ be the restriction of F to M.

One now uses Sard's Theorem for F, which requires F C^2, which we may not have. Let us just assume we have Sard somehow for C^1 F. Let $\overline{a}$ be a regular value for F, and consider the compact curve $\Gamma_a = F^{-1}(\overline{a})$. Γ_a is a submanifold of $\mathbb{R}^n$.

As in [K91], one now defines a vector field ξ in $\mathbb{R}^{n+1}$ by: The derivative Φ_ξ of a C^1 function on $\mathbb{R}^{n+1}$ along ξ is the Jacobian of (F, Φ). Then Γ_a is tangent to ξ, and ξ does not vanish on Γ_a.

Let $g, j, \hat{\jmath}$ be the restrictions of G, J and $\hat{J}$ to Γ_a. So $g'_\xi = j$, and $g'_\xi = \hat{\jmath}$ at the zeroes of g. Now impose the condition that $\tilde{F}$ has its nonregular values of measure zero, but this is automatic since $\tilde{F}$ is C^1, and $\tilde{F}$ is between two manifolds of dimension n. So we can, as Khovanski does, at the cost of discarding elements in a set of measure 0, assume $\overline{a}$ is a regular value for $\tilde{F}$. So g'_ξ does not vanish at the zeroes of g.

Now one attempts induction by considering the systems, based on a shorter chain:

$$\left.\begin{array}{rcl} F & = & \bar{\alpha} \\ \hat{\jmath} & = & \beta \end{array}\right\} \quad \text{for arbitrary } \bar{\alpha}, \beta \text{ (as parameters).}$$

If N is a bound for the number of regular points in $(F, \hat{\jmath})^{-1}(\overline{\alpha}, \beta)$, it follows as in [K91] that for any $\gamma \in \mathbb{R}$, there are at most N regular points in $(\hat{\jmath})^{-1}(\gamma)$. Then, as in [K91] we get that g has $\leq N$ zeroes. So there are $\leq N$ points where $F = \overline{\alpha}, G = 0$. So $(\tilde{F})^{-1}(\overline{\alpha})$ has $\leq N$ points. Applying Sard to $\tilde{F}$ (no extra assumption needed), we get that for any $\overline{\alpha}$, the set set of points M regular for $\tilde{F}$ and in $\tilde{F}^{-1}(\overline{\alpha})$ has cardinal $\leq N$. And this of course gives a bound for the original problem $Q_1, \ldots, Q_n$.

But notice that the argument in the last paragraph certainly needs $\hat{\jmath}$ to be C^1. And we only assumed enough to get $\hat{\jmath}$ continous.

So let us take stock. The intention would be to repeat the argument until we get down to a system in S. It is clear that if we have a finiteness theorem in S then we get one for the Q_i system, *provided* we make enough differentiability assumptions on the Q_i and the P_{ij}. The obvious requirement, taking into account the need to reduce to the proper case, and the C^2 requirement on F early in the discussion, is that the Q_i should be C^{k+3}, and the P_{ij} C^{k+2}. This will of course force the f_i to be C^{k+3}. We have sketched a proof of the following:

Lemma 17: (Hypothesis as above) If for each C^1 $H : \mathbb{R}^m \to \mathbb{R}^m$ (all m) in S, one has a bound (uniformly in parameters) for the number of H-regular points in $H^{-1}(\overline{\alpha})$, then for any system as above, provided the Q_i are C^{k+3} and the P_{ij} C^{k+2}, one also has a uniform bound.

Note that we do not assume the f_i are C^∞, far less real analytic.

5.4 The step from Khovanski Finiteness for systems $H : \mathbb{R}^n \to \mathbb{R}^n$, to the uniform bounds on connected components of zerosets, goes via Morse theory. See, for example, [H76], [T86].

Suppose we have functions $Q_i(\overline{x}, f_0(\overline{x}), \ldots, f_{k-1}(\overline{x}))$ as before, but now $1 \le i \le m$ say. We consider

$$X = \bigcap_{1\le i\le m} Zer\left(Q_i\left(\overline{x}, f_0(\overline{x}), \ldots, f_{k-1}(\overline{x})\right)\right) \subseteq \mathbb{R}^n$$

By taking the sums of the squares of the Q_i, we can assume $m = 1$. We can also assume, at the cost of going into $\mathbb{R}^{n+1}$, that $Q_1(\overline{x}, f_0(\overline{x}), \ldots, f_{k-1}(\overline{x}))$ is proper. Now a general topology argument shows that one can bound the number of connected components for X if one can obtain a uniform bound for the number of components of the zeroset of $Q_1 - \varepsilon$, for ε near 0. Now one wants to apply Sard to Q_1 to get the nonregular ε to form a set of measure 0. For this, unfortunately, we require $Q_1(\overline{x}, f_0(\overline{x}), \ldots, f_{k-1}(\overline{x}))$ to be C^n. Assume this, and replace Q_1 by $Q_1 - \epsilon$ for a regular value ϵ in the image of Q_1 (always possible unless $X = \mathbb{R}^n$). Then X is a manifold, and one easily uses the Morse inequalities [H76] (against a linear Morse function) to bound the number of connected components (and even the sum of Betti numbers) by an upper bound for the number of nonsingular solutions of

$$\begin{aligned} F &= 0 \\ \frac{\partial F}{\partial x_{i_1}} &= 0 \\ &\vdots \\ \frac{\partial F}{\partial x_{i_{n-1}}} &= 0 \end{aligned}$$

where $F(x_1, \ldots, x_n) = Q_1(\overline{x}, f_0(\overline{x}), \ldots, f_{k-1}(\overline{x}))$ and the i_r are distinct with $1 \le i_r \le n$.

But we are therefore reduced to systems already considered in Lemma 17. After some trivial bookkeeping, we get:

Lemma 18: (Hypotheses on the f_i and P_{ij} as in Lemma 17). Suppose that S has the uniform H_0-bounds property. Then the same is true for all sets defined in $\mathbb{R}^n$ by systems $Q_i(\overline{x}, f_0(\overline{x}), \ldots, f_{k-1}(\overline{x})) = 0$ $(1 \leq i \leq m)$, provided

(i) the P_{ij} are C^{k+2};

(ii) the f_i are C^l, where $l = \max(n+1, k+4)$;

(iii) the Q_i are in S, and also C^l.

Again we stress that the f_i are not assumed C^∞. Of course they are if the P_{ij} are (and **(i)** above gives the f_i C^{k+3}).

Lemmas 17 and 18 seem us to give the maximum one can extract from a naive use of Khovanski's method. Now we try to squeeze out more, by assuming more about S.

5.5 Now we begin with an o-minimal Tarski system $\mathcal{S}$ in $\mathbb{R}$, and let S be the system of all C^1 total functions in $\mathcal{S}$. Let $\langle f_0, \ldots, f_{k-1}\rangle$ be a Pfaffian chain over $\mathcal{S}$, the f_i being functions on $\mathbb{R}^n$. Let P_{ij} be the usual functions from S occuring in the differential equations of the chain.

We want to show that the Tarski system got by adding $f_0, \ldots, f_{k-1}$ to $\mathcal{S}$ is o-minimal, and we are going to use the characterization given by Theorem 14. So we have to show, for the system built from S and the f_i, a uniform H_0-condition, and a zeroset condition. For the latter purpose we are going to assume henceforward:

Each f_i is C^∞.

[We do expect that our methods can be elaborated to remove this assumption. A special case is considered in [KM97c]. Patrick Speissegger has informed us in May '97 that by using the quite involved ideas of Lion and Rollin [LR96] he can obtain a considerable generalization of our result [S97].]

We have first to get the uniform H_0-property for sets in $\tilde{\tilde{y}}$ space by conditions (in parameters $\tilde{w}$) defined by positive Boolean combinations of conditions

$$Q\left(\tilde{\tilde{y}}, \tilde{w}, \ldots, \sigma_l^* f_i(\tilde{\tilde{y}}, \tilde{w}), \ldots\right) = 0$$

where Q is in S, and the σ_l^* are various maps on finite sets (cf. 1.3). By going to more variables, we can assume no $>$ condition occurs. Again, the various $\sigma_l^* f_i$ fit into a Pfaffian chain in more variables, so we can assume $\overline{x} = (\tilde{\tilde{y}}, \tilde{w})$ and we are dealing only with conjunctions of conditions

$$Q\left(\tilde{\tilde{y}}, \tilde{w}, f_0(\tilde{\tilde{y}}, \tilde{w}), \ldots, f_{k-1}(\tilde{\tilde{y}}, \tilde{w})\right) = 0$$

Now, as in Khovanskii [K91], we will proceed by induction on k. In [K91] the induction is for systems of m equations in m unknowns, and nonsingular solutions (which are of course isolated). For any given k one uses the latter result, for k, to deal with the case of components of generated systems with the same k, using Morse Theory.

Here a slightly different path has to be followed, since the Q's and the P_{ij} are not assumed C^∞. This causes some problems in the application of Sard.

So we start with a finite system of equations

$$Q(\tilde{\tilde{y}}, \tilde{w}, f_0(\tilde{\tilde{y}}, \tilde{w}), \ldots, f_{k-1}(\tilde{\tilde{y}}, \tilde{w})) = 0 \qquad (\#)$$

(the Q's in S), and want to bound the number of connected components in $\tilde{\tilde{y}}$ space independent of the value given to $\tilde{w}$. If $k = 0$, we get this from the o-minimality of ζ. We now induct on k.

The Morse argument (which needs a generic linear Morse function) takes place in the category of C^2 manifolds [H76]. Because of the Zero Set Condition, we can, in (#), assume as much differentiability of the Q (but not the P_{ij}!) as the problem requires. So, by the standard argument, one can get bounds for components in (#) provided one can get bounds for nonsingular zeros of systems

$$\left.\begin{array}{l} Q(\tilde{\tilde{y}}, \tilde{w}, f_0(\tilde{\tilde{y}}, \tilde{w}), \ldots, f_{k-1}(\tilde{\tilde{y}}, \tilde{w})) = 0 \\ \frac{\partial Q}{\partial y_1}(\tilde{\tilde{y}}, \tilde{w}, f_0(\tilde{\tilde{y}}, \tilde{w}), \ldots, f_{k-1}(\tilde{\tilde{y}}, \tilde{w})) = 0 \\ \quad\vdots \\ \frac{\partial Q}{\partial y_{m-1}}(\tilde{\tilde{y}}, \tilde{w}, f_0(\tilde{\tilde{y}}, \tilde{w}), \ldots, f_{k-1}(\tilde{\tilde{y}}, \tilde{w})) = 0 \end{array}\right\} \qquad (\#\#)$$

Now (##) has as much differentiability as we need (by our earlier assumption), so we may carry out Khovanskii's analysis straightforwardly, getting down to systems

$$\begin{array}{rcl} F & = & \alpha \\ \hat{\jmath} & = & \beta \end{array} \quad (*)$$

as in 5.3, with the estimate we require bounded by the number of nonsingular solutions of (*).

Now we come to the required modification. F does not involve f_{k-1}, and $\hat{\jmath}$ doesn't either, but $\hat{\jmath}$ involves the P_{ij} (from Chain Rule) which are only C^1. The Khovanskii inductive argument falters here, for nonsingular zeros. However, instead of using a bound for nonsingular zeros of (*), we can use one for a number of connected components of (*), and get this by induction since (*) can be written without f_{k-1}. The essential point is that nonsingular solutions for m equations in m unknowns give singleton components.

So now we take stock.
We have proved:

Lemma 19. Let $\mathcal{S}$ be an o-minimal Tarski system on $\mathbb{R}$, and S the system of all C^1 functions in $\mathcal{S}$. Let $\langle f_0, \dots, f_{k-1} \rangle$ be a Pfaffian chain of C^∞ functions over S. Then all Boolean combinations of zero and positivity sets of functions $Q(\bar{x}, f_o(\bar{x}), \dots, f_{k-1}(\bar{x}))$, Q in S, have the uniform H_0-bound property.

Proof: Done. □

We approach our goal.

Lemma 20. (Same assumptions) The class of all such functions above has the Zero Set Property.

Proof: Clear, since the f_i are C^∞. □

Finally:

Theorem 21: Let $\mathcal{S}$ be an o-minimal Tarski system on $\mathbb{R}$, and S the system of all C^1 functions in $\mathcal{S}$. Let $\langle f_0, \dots, f_{k-1} \rangle$ be a Pfaffian chain over S, and assume each f_i is C^∞. Then the Tarski system generated by $\mathcal{S}$ and the f_i is o-minimal.

Proof: Done. □

5.6 The Pfaffian Closure. Let $\mathcal{S}$ be o-minimal. Let *Pfaff*$(\mathcal{S})$ be the smallest Tarski system extending $\mathcal{S}$ and closed under adding C^∞ Pfaffian functions over C^1 functions.

We call *Pfaff*$(\mathcal{S})$ the Pfaffian closure of $\mathcal{S}$. It is o-minimal, by Theorem 21.

6 Concluding Remarks

6.1 Until Wilkie's recent work, the exponential played a special role among transcendental Pfaffian functions not realanalytic at ∞. (It still does vis-a-vis model completeness). Until recently the system $\mathbb{R}_{an,exp}$ was the largest known *o*-minimal system [DM96, DMM94]. It is known [DMM94] that *Pfaff*($\mathbb{R}_{an,exp}$) is much bigger. But even this is by no means the limit. The important work of van den Dries and Speissegger [DS96], based on typically fundamental work of Tougeron [T94], gave two large proper *o*-minimal expansions of $\mathbb{R}_{an}$, namely:

(1) The system based on generalized convergent power series [DS96];

(2) The system based on infinitely representable functions [S96].

Both are even model-complete (both are polynomially bounded). The relation between **(1)** and **(2)** is not quite clear to us, though we believe they have a common *o*-minimal expansion.

If one adds *exp* to **(1)** one gets the ζ function on $(1, \infty)$ definable, and if one adds *exp* to **(2)** one gets the Γ function on $(1, \infty)$ definable. By our Theorem the Pfaffian closure of both systems is *o*-minimal. (Van den Dries [DS97], in work completed before ours, showed that adding *exp* to a (model-complete) *o*-minimal theory containing the restricted exponential preserves *o*-minimality (and model completeness), so the *o*-minimality of ζ, and of Γ, is due to van den Dries and Speissegger).

We fully expect

(a) that all known *o*-minimal expansions of $\mathbb{R}$ have a common extension;

(b) that essentially new *o*-minimal theories, e.g. superexponential, remain to be discovered.

6.2 Indefinite integrals. If $\mathcal{S}$ is *o*-minimal, and f is a C^∞ function $\mathbb{R} \to \mathbb{R}$, f in $\mathcal{S}$, then $g(t) = \int_0^t f(t)dt$ is clearly in *Pfaff*($\mathcal{S}$), and so *o*-minimal, etc. What if f is in $\mathcal{S}$, but only, say, C^1? In [KM97c] we show that in all cases g can be added to $\mathcal{S}$ to preserve *o*-minimality.

On the other hand, if $h(x, y)$ is C^∞, h in $\mathcal{S}$, we do not know if $j(x, y) = \int_0^x h(x, y)dx$ can be added to $\mathcal{S}$ to preserve *o*-minimality. (We currently doubt it).

6.3 The expansion $\sum n! \left(\frac{1}{x}\right)^n$. Euler considered the function

$$g(x) = \int_0^\infty \frac{e^{-t}dt}{1+xt}$$

which has asymptotic expansion $\sum n! \left(\frac{1}{x}\right)^n$ near $+\infty$. This function is not in the system corresponding to the infinitely representable functions and exp [DS97], but it is obviously in $Pfaff(\mathbb{R})$ (exercise).

6.4 VC - dimension. The results of this paper (combined with [DM96, DMM94, W96, DS96, S96]) provide a huge variety of geometric examples of classes with finite VC-dimension. Effective estimates even in a few cases would be very welcome, and in any case one should investigate the significance of the Charbonel operations for VC classes.

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