Approximability of Combinatorial Optimization Problems on Power Law Networks

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Real world networks are not random, they have very small diameter and they possess a power law distribution of node degrees.
Example:
Protein interactions of
*Arabidopsis Thaliana*
Example:
Protein interactions of *Arabidopsis Thaliana*
Example:
Network of Internet Routers
Example:
Network of *Internet Routers*

$\text{Degree (i)}$ vs. $\text{Frequency (y)}$

- Degree (i) on the x-axis, ranging from $10^0$ to $10^4$
- Frequency (y) on the y-axis, ranging from $10^0$ to $10^4$

The graph shows a power-law distribution, indicating a scale-free network.
Example:
*Airport Network* in the United States
Example: Airport Network in the United States
Example:
*Contact Network of karate club members*
Example:

Contact Network of karate club members

Frequency ($y_i$) vs Degree ($i$)

$10^0$ $10^1$

$10^0$ $10^1$
Uniform random graph vs. power law random graph

Number of nodes $y_i$ having degree $i$: $y_i \sim c \cdot i^{-\beta}$
- Uniform random graph vs. power law random graph
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The study of combinatorial optimization problems on real world networks is motivated by applications.
Example: Dominating Set Problem

Minimum dominating set problem in real world networks:

- Optimal sensor or server placement in wireless mobile networks
- Search for key players or nodes in social networks
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Real world networks display a number of other unique and characteristic topological properties.
Real world networks behave like “Small Worlds”

Existence of bridging links across the network
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Real world networks have large clustering coefficients

Clustering coefficient measures cliquishness
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- $C_v = 0$
- Increasing Clustering Coefficient $C_v$
- $C_v = 1$
- Real world networks have embedded hyperbolic geometries
- Relates to Gromov’s four-point condition for $\delta$-hyperbolicity of a metric space
Real world networks have embedded hyperbolic geometries

Relates to Gromov’s four-point condition for $\delta$-hyperbolicity of a metric space

Spherical: $K > 0$

Euclidean: $K = 0$

Hyperbolic: $K < 0$

Decreasing Curvature $K$
There exists a large number of generating models for power law graphs.
Evolving random model for PLG’s:

- The Preferential Attachment Model (Barabási and Albert, 1999)

After adding $u$, probability that $u$ connects to some vertex $v$:

$$\Pr (\{u, v\}) = \begin{cases} \frac{\deg(v)}{\sum_i \deg(v_i)} - 1 & u \neq v \\ \frac{1}{\sum_i \deg(v_i)} & u = v \end{cases}$$
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Static Degree Sequences

**Static random model for PLG’s:**

- The $\mathcal{G}_{\alpha,\beta}$ Model or ACL Model (Aiello, Chung, and Lu, 2001)

Ensures power-law degree distribution by fixing a degree sequence $(y_1, y_2, \ldots, y_\Delta)$ via two parameters $\alpha, \beta$ and then taking the space of random multigraphs with this degree sequence.
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Definition of the ACL Model $G_{\alpha,\beta}$:

For each $1 \leq i \leq \Delta = \lceil e^{\alpha/\beta} \rceil$,

$$y_i = \begin{cases} 
\lceil \frac{e^{\alpha}}{i^\beta} \rceil & \text{if } i > 1 \text{ or } \sum_{i=1}^{\Delta} \lceil \frac{e^{\alpha}}{i^\beta} \rceil \text{ is even} \\
\lceil e^{\alpha} \rceil + 1 & \text{otherwise}
\end{cases}$$

- $\alpha$ is the logarithm of the network size, $\beta$ is the log-log growth rate.
Definition of the ACL Model $G_{\alpha, \beta}$:

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- $\alpha$ is the logarithm of the network size, $\beta$ is the log-log growth rate.
Number of vertices:

\[ n = \sum_{i=1}^{\Delta} \left\lfloor \frac{e^{\alpha}}{i^\beta} \right\rfloor \approx \begin{cases} 
\zeta(\beta) \, e^\alpha & \text{if } \beta > 1 \\
\alpha \, e^\alpha & \text{if } \beta = 1 \\
\frac{e^{\alpha/\beta}}{1-\beta} & \text{if } 0 < \beta < 1
\end{cases} \]

Number of edges:

\[ m = \frac{1}{2} \sum_{i=1}^{\Delta} i \left\lfloor \frac{e^{\alpha}}{i^\beta} \right\rfloor \approx \begin{cases} 
\frac{1}{2} \zeta(\beta - 1) \, e^\alpha & \text{if } \beta > 2 \\
\frac{1}{4} \alpha \, e^\alpha & \text{if } \beta = 2 \\
\frac{1}{2} \frac{e^{2\alpha/\beta}}{2-\beta} & \text{if } 0 < \beta < 2
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ACL Model for PLG’s

- Number of vertices:

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\frac{1}{2} e^{2\alpha/\beta} \frac{1}{2-\beta} & \text{if } 0 < \beta < 2 
\end{cases} \]
The distribution of graphs $G \in \mathcal{G}_{\alpha,\beta}$ over a sequence $(y_1, y_2, \ldots, y_\Delta)$ or $(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ is generated as follows:
1. Generate set $L$ of $\deg(v)$ distinct copies for each vertex $v \in V(G)$

2. $M := \text{random matching}$ on the elements of $L$

3. For $u, v \in V(G)$ number of edges $\{u, v\}$ equals number of edges $m \in M$ that join copies of $u$ and $v$
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\[ \deg(v) = 1 \quad \deg(v) = 2 \quad \deg(v) = 3 \]
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Overview of Main Results
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Presented here:

- Approximation lower bounds for Minimum Dominating Set (Min-DS) in connected PLG’s
- Approximation upper bounds for Minimum Vertex Cover (Min-VC) in random PLG’s

Techniques:

- Connected Embedding Approximation-Preserving (CEAP) reductions
- Transforming hardness results for bounded occurrence CSP’s and Set Cover
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Techniques:

- LP-relaxation and deterministic rounding algorithm
- Upper and lower bounds on the size of half-integral solutions in random PLG’s
Overview of Results

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Further results:

- Approximation lower bounds for \textsc{Min-VC} in connected PLG’s
- Approximation upper bounds for \textsc{Min-DS} for $\beta > 2$
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Further results:

- Approximation lower bounds for \textsc{Min-VC} in connected PLG’s
- Approximation \textit{upper bounds} for \textsc{Min-DS} for $\beta > 2$
Techniques and Paradigms Used
Lower bound technique:

**CEAP reductions (high level view)**

- Embed bounded occurrence CSP and Set Cover reduction instances $G'$ into PLG $G_{\alpha,\beta} \in \mathcal{G}_{\alpha,\beta}$
- Achieve connectivity with reasonable cut sizes between $G'$ and $G_{\alpha,\beta} \setminus G'$
- Preserve hardness of approximation in the embedding construction
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Method: Bounded degree amplifier graphs
(Berman and Karpinski, 1999)
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Basic Idea:

Replace nodes corresponding to variables by 3-regular amplifier
From bounded occurrence CSP’s to vertex covers:

- Reduce bounded occurrence **HYBRID** (equations with 2 and 3 variables) to **MIN-VC** on degree \( d \) bounded graphs (**d-MIN-VC**)
  - Yields explicit lower bounds of \( \frac{103}{102} \) for \( d = 3 \) and \( \frac{55}{54} \) for \( d = 4, 5 \) (Berman and Karpinski, 2003)
  - For larger \( d \) assuming UGC: \( 2 - (2 + o(1)) \frac{\log \log d}{\log d} \) (Austrin, Khot, and M. Safra, 2009)

- **d-MIN-VC** serves as starting point for our CEAP reduction to **MIN-VC** on PLG’s
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From set covering to dominating sets:

- $G_{U,S}$ instances will serve as starting point for our CEAP reduction to $\text{MIN-DS}$ on PLG’s
Set Cover Paradigm

From set covering to dominating sets:

A Set Cover instance \((U, S)\)

\(G_{U,S}\) instances will serve as starting point for our CEAP reduction to MIN-DS on PLG’s
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A Set Cover instance $(U, S)$

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\(G_{U,S}\) instances will serve as starting point for our CEAP reduction to Min-DS on PLG’s
Approximation Lower Bounds for Minimum Dominating Set on Connected Power Law Graphs
Definition (MIN-DS)

**Input:** A graph $G = (V, E)$

**Output:** A subset $D \subseteq V$ such that for each vertex $v \in V$ either $v \in D$ or $D \cup N(v) \neq \emptyset$

**Objective:** Minimize $|D|$
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Dominating Set

Minimum Dominating Set

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Main Results

Approximation Lower Bounds for Min-DS on PLG’s
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## Approximability on general graphs:

- **Upper bound:** $\ln n$ (Johnson, 1974; Lovász, 1975)
- **Lower bound:** $(1 - o(1)) \ln n$ (Feige, 1998)
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Approximability on PLG’s:

- For all $\beta > 0$, $\mathsf{NP}$-hard on simple disconnected PLG’s (Ferrante, Pandurangan, and Park, 2008)

- For all $\beta > 1$, $\mathsf{APX}$-hard on disconnected power law multigraphs (Shen et al., 2012)

Explicit inapproximability factors for $1 < \beta \leq 2$:

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Open Questions

- Is Minimum Dominating Set \( \text{NP} \)-hard and \( \text{APX} \)-hard on connected PLG’s?
- Can we close the gap between the constant lower bounds on PLG’s and the general logarithmic lower bound?
- Can we extend the results to the range \( \beta \in [0, 1] \)?

Theorem (Gast, Hauptmann, and Karpinski, 2012)

For all \( \beta \in [0 + \varepsilon, 2] \) and \( \varepsilon > 0 \), \( \text{MIN-DS} \) is hard to approximate within \( \Omega(\ln(n) - c_\beta) \) on connected PLG’s.
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For all \( \beta \in [0 + \varepsilon, 2] \) and \( \varepsilon > 0 \), **Min-DS** is hard to approximate within \( \Omega(\ln(n) - c_\beta) \) on connected PLG’s.
Embedding technique (CEAP reduction):
- Map $G_{U,S}$ to $G_{\alpha,\beta}$ via scaling construction connecting to a multigraph wheel $W$
  - Number of edges between $G_{U,S}$ and $W$ is $O(\min\{|G_{U,S}|, |W|\})$
- Vertex set $\Gamma$ separates $G_{U,S}$ from $G_{\alpha,\beta} \setminus G_{U,S}$
  - Hardness on maximal component $G_{U,S}$ is preserved
- Maintain small set $X$ to dominate all vertices in $W$
  - MIN-DS is polynomially solvable on $W$
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The Reduction

$\Gamma \subseteq V_2$

$G_{U,S}$

$W$

$X$
Phase Transitions

Observation

For \( \beta > 2 \), \text{Min-DS} on \( G_{\alpha,\beta} \) PLG’s is in \text{APX}

Analysis of the phase transition:

- Study of functional case \( \beta_f = 2 + \frac{1}{f(n)} \)
  - Hard to approximate within \( \Omega(\ln(n) - c_{\theta}) \) for \( f(n) = \omega(\log(n)) \)
  - In \text{APX} for \( f(n) = o(\log(n)) \) (!)
For \( \beta > 2 \), \( \text{Min-DS} \) on \( G_{\alpha, \beta} \) PLG's is in \( \text{APX} \)

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- Study of functional case $\beta_f = 2 + \frac{1}{f(n)}$
  - Hard to approximate within $\Omega(\ln(n) - c_\beta)$ for $f(n) = \omega(\log(n))$
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Approximation Upper Bounds for Minimum Vertex Cover on Random Power Law Graphs
Definition (Min-VC)

**Input:** A graph \( G = (V, E) \)

**Output:** A subset \( C \subseteq V \) such that each edge \( \{u, v\} \in E \) has at least one endpoint in \( C \)

**Objective:** Minimize \( |C| \)
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## Minimum Vertex Cover Problem

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Approximability on general graphs:

- Upper bound: \( 2 - \Theta(1/\sqrt{\log n}) \) (Karakostas, 2009)
- Lower bounds:
  - \( 2 - \varepsilon \) assuming UGC (Khot and Regev, 2008)
  - 1.3606 assuming \( \mathbb{P} \neq \mathbb{NP} \) (Dinur and S. Safra, 2005)
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Approximability on PLG’s:

Observation
There exists practical evidence that Min-VC is easier to approximate on PLG’s

- The greedy algorithm often outperforms the 2-approximation algorithm (Park and Lee, 2001)
- Achieves average ratios of \( \sim 1.02 \) on real world network topologies (M. O. Da Silva, Gimenez-Lugo, and M. V. G. Da Silva, 2013)
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Open Question

Can we give provable guarantees that $\text{MIN-VC}$ is easier to approximate on PLG’s?

Theorem (Gast and Hauptmann, 2012)

There exists an approximation algorithm for $\text{MIN-VC}$ on random $G_{\alpha,\beta}$ PLG’s with expected approximation ratio

$$\rho \leq 2 - \frac{\zeta(\beta) - 1 - \frac{1}{2^\beta}}{2^\beta \zeta(\beta - 1) \zeta(\beta)}$$
Approximation of Min-VC on PLG’s

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Consider the following LP-Relaxation for Min-VC:
There always exists optimal solution which is half-integral, i.e. \( \forall i : x_i \in \{0, \frac{1}{2}, 1\} \) and \( v_i \in V_0, V_{1/2}, V_1 \), respectively.

A half-integral solution can be computed in polynomial time (using algorithm for Min-VC or Perfect Matching in bipartite graphs).
Half Integral Solutions

\[
\text{minimize } \sum_{i=1}^{n} w_i x_i,
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subject to \(x_i + x_j \geq 1\), for all edges \(e = \{v_i, v_j\}\),
\(x_i \geq 0\), for all vertices \(v_i \in V\)

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Approximation Algorithm

Start with **half-integral** solution $x : V \rightarrow \{0, 1/2, 1\}$
Approximation Algorithm

Start with half-integral solution \( x : V \rightarrow \{0, 1/2, 1\} \)

Apply new deterministic rounding algorithm to \( x \)

Prove that algorithm achieves ratio of \( \frac{3}{2} \) on subset \( V' \subseteq V \) of low-degree vertices and their neighborhood.

Overall approximation ratio as convex combination of ratio \( \frac{3}{2} \) on \( V' \) and ratio \( 2 \) on \( V \setminus V' \).

Prove lower bounds on \( x(V') \) and upper bounds on \( x(V) \) to determine the effect of the rounding on global solution.
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Main Results
Approximation Upper Bounds for Min-VC on PLG’s
Approximation Algorithm

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Main Results
Approximation Upper Bounds for Min-VC on PLG’s
Open Problems and Further Research

- Still **improving** on the presented results
  - Investigating the gap between upper and lower approximation bound for MIN-VC on PLG’s
  - Improving upper bounds for MIN-DS on PLG’s when $\beta \leq 2$ (in random or quasi-random models)

- Exploit network hyperbolicity in biological and Internet based network design problems

- Computational complexity of node and edge deletion problems and information spreading in dynamic networks (especially in biological settings)

- Applicability of graph limit theory in order to gather topological information of PLG generating processes
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Thank you!