2. The price of anarchy for nonatomic routing games

References:

- T. Roughgarden, É. Tardos, How bad is Selfish Routing?, JACM 49, 236-259 (2002).
- T. Roughgarden, The Price of Anarchy is Independent of the Network Topology, JCSS 67, 341-364 (2003).
- R. Feldmann, M. Gairing, T. Lücking, B. Monien, M. Rode, Selfish Routing in Non-cooperative Networks: A Survey, MFCS 2003, LNCS 2747, 21-45 (2003).

Networks with noncooperating users without central control can be modelled using non-cooperative games. The Internet is a popular example for such a network. The goal of every user is to minimize his own cost without respect to the cost of the other users. Often this contradicts the social value of the solution. Note that the social value might be a measure for the global value of the solution

Question:
What is the price for the missing of a central control? This means: What is the price of anarchy?

The usual measure for the price of anarchy is the socalled <u>coordination ratio</u>. This is the ratio of the social values of a worst case Nash equilibrium and an optimal solution.

For the nonatomic routing game the social value $C(f)$ of a feasible solution f is defined by

$$C(f) := \sum_{e\in E} f(e) \cdot d_e(f(e)).$$

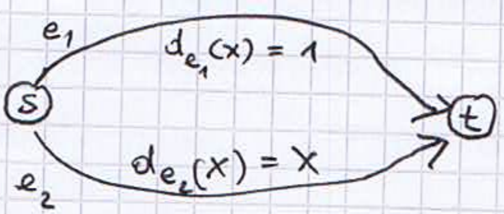
We shall investigate the coordination ratio of nonatomic routing games.

<u>Question:</u>

What is the maximal possible value of the coordination ratio?

Using an example we shall show that the coordination ratio can be arbitriary large.

<u>Example 2.1</u> (Pigou 1920)



A flow of size one is to send from s to t.

Sending the whole flow on edge e_2 is a Nash equilibrium. For this flow f we obtain

$$C(f) = 0 \cdot 1 + 1 \cdot 1 = 1$$

Theorem 1.20 $\Rightarrow$

A flow f^* is optimal iff

$$d_{e_1}(f^*(e_1)) + f^*(e_1) \cdot d'_{e_1}(f^*(e_1))$$
$$= d_{e_2}(f^*(e_2)) + f^*(e_2) \cdot d'_{e_2}(f^*(e_2))$$

Note that

$$d_{e_1}(f^*(e_1)) + f^*(e_1)\, d'_{e_1}(f^*(e_1)) = 1 + f^*(e_1) \cdot 0 = 1$$

and

$$d_{e_2}(f^*(e_2)) + f^*(e_2) \cdot d'_{e_2}(f^*(e_2)) = f^*(e_2) + f^*(e_2) \cdot 1$$
$$= 2 \cdot f^*(e_2)$$

Hence, f^* is optimal iff

$$f^*(e_1) = f^*(e_2) = \frac{1}{2}$$

For this flow we obtain

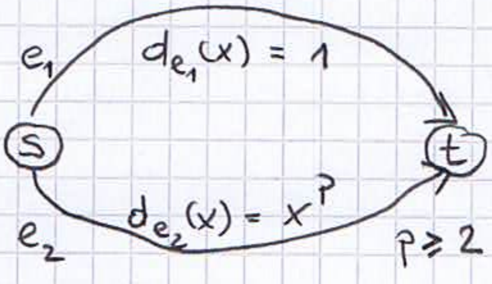
$$C(f^*) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = 3/4.$$

Therefore, we obtain the following coordination ratio:

$$\frac{C(f)}{C(f^*)} = \frac{1}{3/4} = \frac{4}{3}.$$

We can modify Pigou's example such that the coordination ratio becomes arbitrary large.

Example 2.2



A flow of size one has to be sent from s to t.

For all $p \geq 2$, we obtain a Nash equilibrium by sending the whole flow f on the edge e_2. For this flow f we obtain

$$C(f) = 0 \cdot 1 + 1 \cdot 1 = 1$$

Theorem 1.20 $\Rightarrow$

A flow f^* is optimal iff

$$d_{e_1}(f^*(e_1)) + f^*(e_1)\, d_{e_1}'(f^*(e_1)) = d_{e_2}(f^*(e_2)) + f^*(e_2)\, d_{e_2}'(f^*(e_2)).$$

Note that

$$d_{e_1}(f^*(e_1)) + f^*(e_1)\, d_{e_1}'(f^*(e_1)) = 1$$

and

$$d_{e_2}(f^*(e_2)) + f^*(e_2)\, d_{e_2}'(f^*(e_2)) = f^*(e_2)^p + f^*(e_2) \cdot p \cdot f^*(e_2)^{p-1}$$

Hence, f^* is optimal iff

$$f^*(e_2)^p + f^*(e_2)\, p \cdot f^*(e_2)^{p-1} = 1$$

$$\Leftrightarrow \quad (p+1)\, f^*(e_2)^p = 1$$

$$\Leftrightarrow \quad f^*(e_2) = (p+1)^{-\frac{1}{p}}$$

For this flow we obtain

$$\begin{aligned} C(f^*) &= \left(1-(p+1)^{-\frac{1}{p}}\right)\cdot 1 + (p+1)^{-\frac{1}{p}} \cdot \left((p+1)^{-\frac{1}{p}}\right)^p \\ &= 1 - (p+1)^{-\frac{1}{p}} + (p+1)^{-1} \cdot (p+1)^{-\frac{1}{p}} \\ &= 1 - \left(1 - \frac{1}{p+1}\right) \cdot (p+1)^{-\frac{1}{p}} \end{aligned}$$

Hence,

$$C(f^*) \xrightarrow[p\to\infty]{} 0$$

Therefore, the coordination ratio $\frac{C(f)}{C(f^*)}$ converges to ∞ if $p \to \infty$.

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<u>Question</u>:

In dependence on the possible delay functions does there exist always a "simple" graph G such that the worst case coordination ratio is obtained by a routing game on G?

The following theorem gives us an answer to this question.

Theorem 2.1

Let $\mathcal{D}$ be a class of delay functions such that

i) $\mathcal{D}$ contains the constant functions and
ii) $x \cdot d_e(x)$ is convex and continuously differentiable for all functions d_e in $\mathcal{D}$.

Then the worst case of the coordination ratio is obtained by a routing game on a graph $\bar{G} = (\bar{V}, \bar{E})$ where $|\bar{V}| = |\bar{E}| = 2$ and one edge has a constant delay function.

Proof:

Idea:

Start with any routing game R_{wc} with delay functions in $\mathcal{D}$ such that its coordination ratio is the worst for the class $\mathcal{D}$. Using this routing game construct a routing game on a graph $\bar{G} = (\bar{V}, \bar{E})$ with $|\bar{V}| = |\bar{E}| = 2$ such that its coordination ratio cannot be better than the coordination ratio of R_{wc}.

Realization:

Let $G = (V, E)$ be the graph of the routing game R_{wc} and let $\tilde{f}$ be a worst case Nash equilibrium with respect to the game R_{wc}.

Let $G' = (V, E')$ be the graph which we obtain from G by adding to each edge $e \in E$ a parallel copy e'; i.e.,

$$E' = \{ e, e' \mid e \in E \}.$$

In G', each edge $e \in E$ has the same delay function d_e as in G. The copy e' of the edge $e \in E$ obtains the constant delay function

$$d_{e'}(x) := d_e(\tilde{f}(e)).$$

In G', the same flow as in G has to be sent from the source s_i to the sink t_i, $1 \leq i \leq k$.

Construction $\Rightarrow$

The optimal flow of the routing game on G' cannot be worse than an optimal flow of the routing game on G.

Let $\bar{f} : E' \to \mathbb{R}$ be defined by

$$\bar{f}(e) = \begin{cases} \tilde{f}(e) & \text{if } e \in E \\ 0 & \text{if } e \in E' \setminus E \end{cases}$$

Construction $\Rightarrow$

$\bar{f}$ is a Nash equilibrium of the routing game on G'.

Claim:

The computation of an optimal flow for the routing game on G_i.

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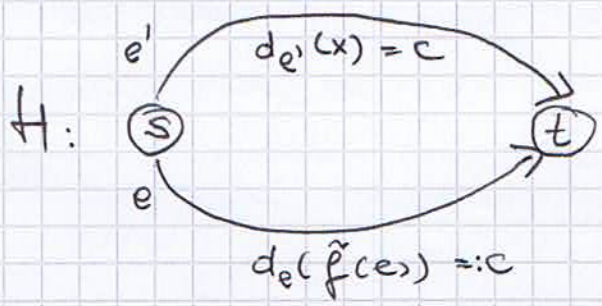
Let us consider the following situation:



A flow of size $\tilde{f}(e)$ should be sent from s to t.

Goal:

Determination of an optimal flow in H.

The social value $C(\tilde{f})$ of $\tilde{f}$ is the following:

$$C(\tilde{f}) = 0 \cdot c + \tilde{f}(e) \cdot d_e(\tilde{f}(e)) = \tilde{f}(e) \cdot c$$

Theorem 1.20 $\Rightarrow$

A flow f^* is optimal iff

$$d_{e'}(f^*(e')) + f^*(e') \cdot d'_{e'}(f^*(e')) = d_e(f^*(e)) + f^*(e) \cdot d'_e(f^*(e)).$$

Since $d'_{e'}(f^*(e')) = 0$ we obtain

$$d_{e'}(f^*(e')) + f^*(e') \cdot d'_{e'}(f^*(e')) = c.$$

Let $f^*(e)$ be the flow on e in an optimal flow.
Then

Before we continue the proof of the theorem we consider our definition of a Nash equilibrium for nonatomic routing games again. We have written

" A feasible flow is a Nash equilibrium if no player can improve the own situation by diversion his flow; i.e.,

$$\forall\ i \in \{1,2,\dots,k\}\ \forall\ P, Q \in \mathcal{P}_i\ \forall\ \delta \in (0, f_P)$$

$$d_P(f) \leq d_Q(\tilde{f})$$

where

$$\tilde{f}_R := \begin{cases} f_R - \delta & \text{if } R = P \\ f_R + \delta & \text{if } R = Q \\ f_R & \text{if } R \notin \{P, Q\} \end{cases}$$

$\tilde{f}$ is obtained by the diversion of δ flow from P to Q".

The definition is taken from

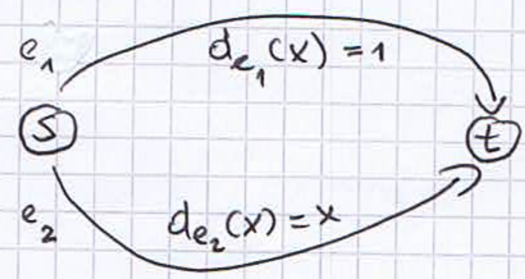
Tim Roughgarden, Éva Tardos: "How Bad Is Selfish Routing?", JACM 49, pp 236-259, 2002.

They write (p. 242):

" We wish to consider flows that represent an equilibrium among many noncooperative agents – that is, flows that behave "greedily" and "selfishly," without regard to the overall cost. Intuitively, we expect each unit of such a flow (no matter how small) to travel along the minimum-latency path available to it, where latency is measured with respect to the rest of the flow; otherwise, this flow would re-route itself on a path with smaller latency. We formalize this idea in the next definition."

In the context of Roughgarden and Tardos, each unit of flow is a player which wish to obtimize the own solution.

If the players are the k source/sink-pairs, the definition above does not define a Nash equilibrium. For seeing this let us consider Pigou's example again.



A flow of size one has to be sent from s to t.

In the sence of the definition above, the whole flow on edge e_2 would be a Nash equilibrium.

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If we have one player i which has to send the flow of size one from s to t, the question what is a Nash equilibrium depends on the cost function which the player i wish to minimize

Possible cost functions:

a) $\sum_{P \in \mathcal{P}_i} f_P \cdot d_P(f)$

b) $\max_{P \in \mathcal{P}_i} f_P \cdot d_P(f)$

If the player i minimizes the first cost function then a Nash equilibrium f^* would be

$$f^*(e_1) = f^*(e_2) = 1/2$$

If the player i minimizes $\max_{P \in \mathcal{P}_i} f_P \, d_P(f)$ then a Nash equilibrium f^* would be

$$f^*(e_1) = 1 - f^*(e_2)$$

and $f^*(e_2)$ is the solution of

$$(f^*(e_2))^2 + f^*(e_2) = 1.$$

We use the definition of Roughgarden and Tardos.

$$\tilde{f}(e) - f^*(e) = f^*(e')$$

is the flow on e' in an optimal flow f^*.

We can compute $f^*(e)$ by solving the equation

$$d_e(f^*(e)) + f^*(e) \cdot d_e'(f^*(e)) = c.$$

Claim:

From the flow $\bar{f}$ on G' we can compute an optimal flow f^* on G' by dividing for each edge/copy-pair the flow $\bar{f}(e)$ as described above.

Proof of claim:

Consider the following delay functions d_e^*:

$$d_e^*(x) := d_e(x) + x\, d_e'(x).$$

An optimal distribution f^* of the flow $\bar{f}(e)$ to the pair $e, e' \in E'$ implies

$$d_e^*(f^*(e)) = d_{e'}^*(f^*(e')) = c = d_e(\bar{f}(e)).$$

Our goal is to prove that f^* is a Nash equilibrium with respect to the delay functions d_e^*, $e \in E'$.

Assume that f^* is not a Nash equilibrium with respect to the delay functions d_e^*.

Lemma 1.2 $\Longrightarrow$

$\exists\, i \in \{1, 2, \ldots, k\}$ $\exists\, P \in \mathcal{P}_i$ with $f^*_P > 0$
$\exists\, Q \in \mathcal{P}_i \setminus \{P\}$ such that

$$d^*_P(f^*) > d^*_Q(f^*).$$

This means

$$(*) \quad \sum_{e \in P} d^*_e(f^*(e)) > \sum_{e \in Q} d^*_e(f^*(e))$$

$\Longleftrightarrow$

$$\sum_{e \in P} d_e(f^*(e)) + f^*(e)\, d'_e(f^*(e))$$

$$> \sum_{e \in Q} d_e(f^*(e)) + f^*(e)\, d'_e(f^*(e))$$

For $R \in \mathcal{P}_i$ let $\hat{R}$ be that path from s_i to t_i which we obtain from R if we choose from the edge/copy pair $\{e, e'\}$ with $\{e, e'\} \cap R \neq \emptyset$ always the edge e'.

The construction of f^* and the definition of $d^*_{e'}$

$\Longrightarrow$

$$d^*_{\hat{R}}(f^*) = d^*_R(f^*) \quad \forall\, R \in \mathcal{P}_i$$

Hence, by $(*)$

$$\sum_{e' \in \hat{P}} d^*_{e'}(f^*(e')) > \sum_{e' \in \hat{Q}} d^*_{e'}(f^*(e'))$$

Note that

$$d_{e'}(x) = \underbrace{d_e(\tilde{f}(e))}_{\text{constant.}} \quad \forall\, e' \in \hat{P} \cup \hat{Q}$$

$\Longrightarrow$

$$\sum_{e' \in \hat{P}} d^*_{e'}(f^*(e')) = \sum_{e' \in \hat{P}} d_{e'}(f^*(e')) + f^*(e') \underbrace{d'_{e'}(f^*(e'))}_{=0}$$

$$= \sum_{e' \in \hat{P}} d_e(\tilde{f}(e))$$

$$= \sum_{e \in P} d_e(\tilde{f}(e))$$

In the same way we obtain

$$\sum_{e' \in \hat{Q}} d^*_{e'}(f^*(e')) = \sum_{e \in Q} d_e(\tilde{f}(e))$$

Therefore, we obtain

$$\sum_{e \in P} d_e(\tilde{f}(e)) > \sum_{e \in Q} d_e(\tilde{f}(e))$$

This is a contradiction to the fact that $\tilde{f}$ is a Nash equilibrium in $G = (V, E)$ with respect to the delay functions d_e, $e \in E$.

$\Rightarrow$

The assumption is false. Hence, f^* is a Nash equilibrium with respect to the delay functions d^*_e, $e \in E'$.

$\Longrightarrow$ by Corollary 1.1 $\quad f^*$ is an optimal flow in G' $\square$

Hence, we obtain

$$\frac{C(\bar{f})}{C(f^*)} = \frac{\sum_{e\in E'} \bar{f}(e) \cdot d_e(\bar{f}(e))}{\sum_{e\in E'} f^*(e) \cdot d_e(f^*(e))}$$

$$= \frac{\sum_{e\in E} \bar{f}(e) \cdot d_e(\bar{f}(e))}{\sum_{e\in E'} f^*(e) \cdot d_e(f^*(e))}$$

Exercise:
Prove $\frac{a+b}{a'+b'} \leq \max\left\{\frac{a}{a'}, \frac{b}{b'}\right\}$ for $0 \leq a, b$, $0 < a', b'$

$$\leq \max_{e\in E} \frac{\bar{f}(e) \cdot d_e(\bar{f}(e))}{f^*(e)\, d_e(f^*(e)) + f^*(e')\, d_{e'}(f^*(e'))}$$

Let $e_0 = (v, w)$ be an edge which maximize the right side of the inequality above.

Choose $\bar{G} = (\bar{V}, \bar{E})$ where

$$\bar{V} = \{v, w\}, \quad \bar{E} = \{e, e'\} \text{ such that}$$

$e = (v, w)$ and e' is the copy of e.

The delay functions d_e and $d_{e'}$ are the same as in G'. A flow of size $\bar{f}(e)$ has to be sent from v to w.

As proved above, the worst case of the coordination ratio with respect to the class D of delay functions is obtained by the routing game defined for $\bar{G}$. This proves the theorem ■