8. Markets and market equilibria

8.1 A simple market game

Given

- n sellers of a product (for example ice cream)
- scenario

The buyers are equally distributed on the line segment $[0,1] \subset \mathbb{R}$. Each of them buys the ice cream at a nearest seller. The buyers are equally distributed to the nearest sellers.

The intention of each seller is to choose his location on $[0,1]$ such that his gain is maximized.

We shall discuss the Nash equilibria for the cases $n \in \{1,2,3\}$.

$n = 1$:

The only seller has a monopoly. Hence, he can choose any point on $[0,1]$ to obtain a Nash equilibrium.

$n = 2$:

If the shops of both sellers are not placed at the same point each seller can improve his gain if he places his shop closer to the shop of the other seller.

$\Rightarrow$

In a Nash equilibrium, both shops are placed at the same point.

If the shops are placed at a point $\neq 0.5$ then each player can enlarge his gain by placing his shop closer to 0.5.

$\Rightarrow$
The only Nash equilibrium is both shops are placed at the point 0.5.

<u>$n = 3$</u>

We distinguish three cases:

<u>Case 1</u>: The three shops are not placed at the same point.

Let us consider a single shop which is not placed between the other two shops. It is clear that such a shop exists.

This shop can improve his gain by choosing a point which is closer to the other two shops.

$\Rightarrow$ no Nash equilibrium.

<u>Case 2</u>: The three shops are placed at the same point $\neq 0.5$.

Then each seller can enlarge his gain if he places his shop at the point 0.5. Instead of a third of the buyers he would serve at least the half of the buyers.

$\Rightarrow$ no Nash equilibrium.

Case 3: The three shops are placed at the point 0.5.

Then every of the three sellers can enlarge his gain if he place his shop a little bit away from the point 0.5. Then instead a third of the buyers he would serve almost the half of the buyers.

$\Rightarrow$ no Nash equilibrium.

Altogether we have shown that in the case $n=3$ no pure Nash equilibrium exists.

Exercise:
For $n = 3$ does there exists a mixed Nash equilibrium? What is the situation for $n > 3$?

Implicitely, we have assumed that the prizes of the ice cream is the same for all sellers.

Assume that each seller can determine his own price for his ice cream. Then among other things the following problems arise:

- Possibly, for the buyers there is a tradeoff between the price of the product and the closeness of the shop.

- The quantity buyed by a customer can depend on the price of the product.

What is the social value of a solution of the game above? Possibly, we can consider the ratio

$$\frac{\text{quantity}}{\text{price}}$$

$\Rightarrow$

A seller has to find a balance between the maximation of the profit and the minimization of the price.

$\rightsquigarrow$

Goal:

The development of a model for a market game with prices.

8.2 A market game with prices

Given

- n markets $M = \{m_1, m_2, \dots, m_n\}$
- k sellers $A = \{a_1, a_2, \dots, a_k\}$
- L_j, $1 \leq j \leq k$ set of possible locations of the seller a_j.
- bipartite weighted graph

$$G = (M, L, E, c)$$

such that

$$L = \bigcup_{j=1}^{k} L_j \,,$$

$$E = \{ (i,\ell) \mid 1 \le i \le n,\ \ell \in L \} \,,$$

$c_{i\ell}$, $(i,\ell) \in E$ cost for serving the market m_i from the location ℓ

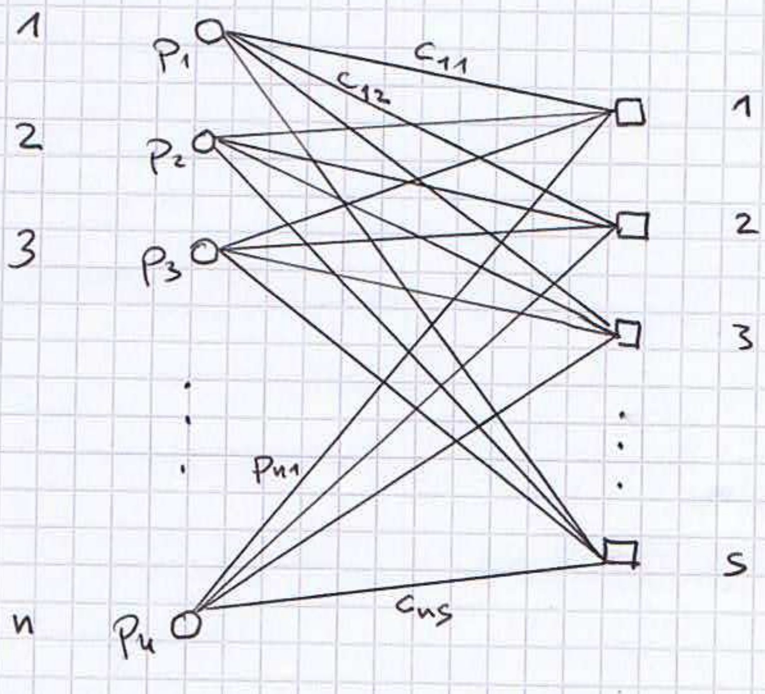
- p_i, $1 \le i \le n$ the maximal price which can be payed by the market m_i.

Let $L = \{\ell_1, \ell_2, \ldots, \ell_s\}$. Then the edge set E can be defined as follows:

$$E := \{ (i,j) \mid 1 \le i \le n,\ 1 \le j \le s \}.$$

$\rightsquigarrow$

markets



If a market m_r cannot be served from a location ℓ_t then this can be modelled by setting $c_{rt} = \infty$.

For the special case $c_{ij} \neq c_{is}$ for $j \neq s$ we shall prove that the market game with prices is a potential game.

Notations:

Let $L' \subseteq L$ be the set of locations which have a seller.

π_{ij}, $1 \leq i \leq n$ denotes the price of the seller at location $l_j \in L'$ for the market m_i.

Analysis

a) The point of view of the markets:

Each market m_i buys his goods from the cheapest seller. Let $\sigma(i)$ denote the location of the cheapest seller for the market m_i if such a seller exists. I.e.,

$$\sigma(i) = \begin{cases} j_0 \text{ where } l_{j_0} \in L' \text{ and} & \\ \quad \pi_{ij_0} = \min\limits_{l_k \in L'} \pi_{ik} \leq p_i & \text{if } j_0 \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases}$$

b) The point of view of the sellers:

The seller at location $l_j \in L'$ attempts to hold his price π_{ij} with respect to the market m_i below the prices of the other sellers. This means that π_{ij} should be below π_{ik}, $k \neq j$, $l_k \in L'$. Furthermore, his gain should be positive, i.e.,

$$\pi_{ij} > c_{ij} .$$

$\Rightarrow$

The seller at location $l_j \in L'$ can offer a price below all other sellers iff for all $l_k \in L' \setminus \{l_j\}$ there holds

$$c_{ij} < c_{ik} .$$

$\Rightarrow$

It holds also

$$\sigma(i) = \begin{cases} j_0 \text{ where } l_{j_0} \in L' \text{ and} & \\ \quad c_{ij_0} = \min\limits_{l_k \in L'} c_{ik} < p_i & \text{if } j_0 \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Determination of the prices:

The seller at location $\sigma(i)$ chooses the second smallest value c_{ik} with $l_k \in L'$ or p_i in dependence if $\min\{c_{ik} \mid l_k \in L' \setminus \{l_{\sigma(i)}\}\} \leq p_i$ or not. $\rightsquigarrow$

$$\pi_{i\sigma(i)} := \begin{cases} \min\{p_i, \min\{c_{ik} \mid l_k \in L' \setminus \{l_{\sigma(i)}\}\}\} & \\ & \text{if } c_{i\sigma(i)} < p_i \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Theorem 8.1

If for all i for all j, k with $j \neq k$, $c_{ij} < \infty$ and $c_{ik} < \infty$ also $c_{ij} \neq c_{ik}$ then the market game with prices is a potential game.

Proof:

Let $M' \subseteq M$ be the set of markets m_i such that there is a seller which offers a price $\leq p_i$.
We define the potential function Φ in the following way:

$$\Phi := \sum_{m_i \in M'} c_{i\sigma(i)} + \sum_{m_i \notin M'} p_i$$

Interpretation:

The value of the potential function corresponds to the quality of the solution with respect to the social value.

Definition $\Rightarrow$

To prove that Φ is indeed a potential function it suffices to show the following:

If a single seller changes his location then the difference of the values of Φ is equal to the change of the profit of the seller which changes his location.

Assume that the seller a_c changes from location l_j to location l_k.

Let

Φ the potential function before the change
Φ' the potential function after the change
$\sigma'(i)$ the location of the cheapest seller for the market m_i after the change, $1 \le i \le n$.

Agreement:

If $m_i \notin M'$ then $c_{i\sigma(i)}$ and $c_{i\sigma'(i)}$, respectively denotes the price p_i.

We partition the set M of markets into the following four pairwise disjoint sets:

$S_1 := \{ i \mid \sigma(i) = j \text{ and } \sigma'(i) = k \}$,

$S_2 := \{ i \mid \sigma(i) = j \text{ and } \sigma'(i) \neq k \}$

$S_3 := \{ i \mid \sigma(i) \neq j \text{ and } \sigma'(i) = k \}$

$S_4 := \{ i \mid \sigma(i) \neq j \text{ and } \sigma'(i) \neq k \}$

P_i, $1 \le i \le 4$ denotes the change of the profit of the seller a_c with respect to the markets in S_i.

For each market $i \in S_1$ there holds:

- Before the change of the location:
  $l_k \notin L'$ and Π_{ij} does not depend on l_j.

- After the change of the location:

$$\ell_j \notin L' \text{ and } \Pi_{ik} \text{ does not depend on } \ell_k.$$

Hence, it follows from the definition

$$\Pi_{ik} = \Pi_{ij}$$

Therefore, we obtain the following change of the profit with respect to the market m_i:

$$(\Pi_{ik} - c_{i\sigma'(i)}) - (\Pi_{ij} - c_{i\sigma(i)})$$

$$= c_{i\sigma(i)} - c_{i\sigma'(i)}$$

Hence,

- $$P_1 = \sum_{i \in S_1} (c_{i\sigma(i)} - c_{i\sigma'(i)})$$

For each market $i \in S_2$ there holds

$$\Pi_{ij} = c_{i\sigma'(i)}.$$

This follows from the fact that the location corresponding to the second smallest cost is the location assigned to the market m_i after the change of location.

Hence, the profit of the seller a_c is reduced by

$$- (\Pi_{ij} - c_{i\sigma(i)})$$

$$= - (c_{i\sigma'(i)} - c_{i\sigma(i)})$$

$$= c_{i\sigma(i)} - c_{i\sigma'(i)}.$$

Therefore, we obtain

$$P_2 = \sum_{i \in S_2} (c_{i\sigma(i)} - c_{i\sigma'(i)})$$

Analogously, we obtain

- $$P_3 = \sum_{i \in S_3} (c_{i\sigma(i)} - c_{i\sigma'(i)})$$ and

- $$P_4 = \sum_{i \in S_4} (c_{i\sigma(i)} - c_{i\sigma'(i)})$$

Exercise:

Show that P_3 and P_4 above are given correctly.

In total, we obtain the following change of profit of the seller a_c:

$$\begin{aligned} P &= P_1 + P_2 + P_3 + P_4 \\ &= \sum_{i \in M} (c_{i\sigma(i)} - c_{i\sigma'(i)}) \\ &= \Big(\sum_{i \in M'} c_{i\sigma(i)} + \sum_{i \notin M'} p_i \Big) \\ &\quad - \Big(\sum_{i \in M''} c_{i\sigma'(i)} + \sum_{i \notin M''} p_i \Big) \end{aligned}$$

where M'' is the set of markets m_i after the location change such that there is a seller offering a price $\leq p_i$.

This proves that ϕ is indeed a potential function.

Theorem 1.5 $\Rightarrow$

Corollary 8.1

The special case above of the market game with prices has a pure Nash equilibrium.

The social value of a solution is given by the sum of the contributions of all markets. The contribution of a market which is not served by a seller is zero. The contribution of a market m_i served by the seller at location $\sigma(i)$ consists of

- the saving $p_i - \pi_{i\sigma(i)}$ of the market m_i

and

- the profit $\pi_{i\sigma(i)} - c_{i\sigma(i)}$ of the seller.

Hence, the total contribution is

$$p_i - c_{i\sigma(i)}.$$

Therefore, the social value of a solution is

$$\left(\sum_{i=1}^{n} p_i \right) - \phi .$$

Question:

How relates the social value of a pure Nash equilibrium and the social value of an optimal solution?

The following theorem gives an answer to this question.

Theorem 8.2

The social value of an optimal solution of the special case above of the market game with prices is at most the double of the social value of a pure Nash equilibrium.

Proof:

Given any pure Nash equilibrium let

- $j_1, j_2, \ldots, j_k$ be the locations of the sellers in the Nash equilibrium
- $j_1', j_2', \ldots, j_k'$ be the locations of the sellers in the optimal solution
- $\sigma(i)$ the location of the seller assigned to market m_i in the Nash equilibrium
- $\sigma'(i)$ the location of the seller assigned to market m_i in the optimal solution

We identify seller and the index of his location.
Consider the possible change of seller ℓ from location j_ℓ to location j_ℓ'.

Theorem 8.1 $\Rightarrow$

utility of seller ℓ $\hat{=}$ exactly the improvement of the potential function ϕ.

$\Longrightarrow$

This is exactly the improvement of the social value

$$\left(\sum_{i=1}^{n} p_i\right) - \phi.$$

The solution is a pure Nash equilibrium.

$\Longrightarrow$

The change of a single seller does not improve his profit and hence, also not the social value.

<u>But</u>

The change of the location of ≥ 2 sellers could improve the social value.

<u>Goal:</u>

Estimation of the best possible improvement because of such a change of location.

Let

- $val'(e)$ the new profit of the seller e if he change his location alone from j_e to j_e'
- $\delta(i)$ the difference between the contributions of the market m_i in the Nash

equilibrium and the optimal solution.
This means

$$\delta(i) = \begin{cases} c_{i\sigma(i)} - c_{i\sigma'(i)} & \text{if } m_i \text{ is sered in both solutions} \\ p_i - c_{i\sigma'(i)} & \text{if } m_i \text{ is served only in the optimal solution} \\ c_{i\sigma(i)} - p_i & \text{if } m_i \text{ is served only in the Nash equilibrium} \\ 0 & \text{if } m_i \text{ is not sered in both solutions} \end{cases}$$

Let $M'(e)$ be the set of markets which are served by the seller e in the optimal solution.

Observation (*):

There holds $val'(e) \geq \sum_{i \in M'(e)} \delta(i)$

We have equality if all markets $m_i \in M'(e)$ are sered by another seller in the Nash equilibrium and inequality otherwise.

Exercise:

Prove the observation (*).

Let val(e) denote the profit of seller e in the Nash equilibrium.

Properties of a Nash equilibrium $\Rightarrow$

$$\text{val}'(e) \leq \text{val}(e) \quad \text{for all } e$$

$\Rightarrow$

$$\sum_e \text{val}(e) \geq \sum_e \text{val}'(e)$$
$$\geq \sum_e \sum_{i \in M'(e)} \delta(i)$$
$$\geq \sum_i \delta(i)$$

Note that $\sum_i \delta(i)$ is exactly the total improvement of the social value if we change from the Nash equilibrium to the optimal solution.

Let

Φ be the potential function of the Nash equilibrium

and let

Φ' be the potential function of the optimal solution.

Then there holds

$$\left(\sum_{i=1}^{n} p_i\right) - \phi' = \left(\left(\sum_{i=1}^{n} p_i\right) - \phi\right) + \phi - \phi'$$

$$= \left(\left(\sum_{i=1}^{n} p_i\right) - \phi\right) + \sum_{i} \delta(i)$$

$$\leq \underbrace{\left(\left(\sum_{i=1}^{n} p_i\right) - \phi\right)}_{\text{total profit of all sellers + utility of all markets}} + \underbrace{\sum_{e} \mathrm{val}(e)}_{\text{total profit of all sellers}}$$

$$\leq 2\left(\left(\sum_{i=1}^{n} p_i\right) - \phi\right).$$