$$\left(\sum_{i=1}^{n} p_i\right) - \varphi' = \left(\left(\sum_{i=1}^{n} p_i\right) - \varphi\right) + \varphi - \varphi'$$

$$= \left(\left(\sum_{i=1}^{n} p_i\right) - \varphi\right) + \sum_{i} \delta(i)$$

$$\leq \underbrace{\left(\left(\sum_{i=1}^{n} p_i\right) - \varphi\right)}_{\substack{\text{total profit of all}\\ \text{sellers + utility}\\ \text{of all markets}}} + \underbrace{\sum_{e} \mathrm{val}(e)}_{\substack{\text{total profit of}\\ \text{all sellers}}}$$

$$\leq 2\left(\left(\sum_{i=1}^{n} p_i\right) - \varphi\right).$$

■

8.3 Fisher market equilibria

References

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N.R. Devanur, Ch. Papadimitriou, A. Saberi, V. Vazirani, Market equilibrium via a primaldual algorithm for a convex program, JACM (2008), 1 – 18.

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A Fisher market consists of

- a set B of n agents and
- a set A of m divisible goods.

Each agent a_i, $1 \le i \le n$ has an endowment e_i of money and a concave utility function

$$u_i : \mathbb{R}^m_+ \to \mathbb{R}_+ .$$

Let $\bar{x}_i \in \mathbb{R}^m_+$ denote the vector corresponding to the agent a_i. The j-th component x_{ij} of $\bar{x}_i$ denotes exactly the quantity of good j buyed by agent a_i. Then $u_i(\bar{x}_i)$ denotes the utility of agent a_i which he has according to his buyed goods.

Let $\bar{p} = (p_1, p_2, \dots, p_m) \in \mathbb{R}^m_+$ denote the price vector with respect to the m goods; i.e., p_j EURO is the price of one unit of good j, $1 \le j \le m$.

The goal of each agent a_i is to use his endowment e_i for buying a bundle $\bar{x}_i = (x_{i1}, x_{i2}, \dots, x_{im})$ of goods which is optimal with respect to the price vector $\bar{p}$. This means that agent a_i has to solve the following optimization problem:

$$\max\ u_i(\bar{x}_i)$$
$$\sum_{j=1}^{m} p_j x_{ij} \leq e_i.$$

The goal is to find a price vector $\bar{p} = (p_1, p_2, \dots, p_m)$ such that each agent uses his whole endowment, buys an optimal bundle of goods, and for each good the whole quantity finds its buyer. Such a price vector is called a <u>market equilibrium</u>.

The model above has been introduced by Irving Fisher in 1891.

We will consider the special case of linear utility functions. A function u_i is called <u>linear</u> if

$$u_i(\bar{x}_i) = \sum_{j=1}^{m} u_{ij} x_{ij}$$

where $u_{ij} \in \mathbb{R}_+$.

<u>Goal</u>:

The development of an algorithm for the computation of a market equilibrium in Fisher markets with linear utility functions.

<u>Remark</u>:

Using scaling we can normalize the total amount of each good to 1.

Exercise:

Let b_j, $1 \le j \le m$ denote the amount of good j. Define a scaling such that all amounts become to be one and a market equilibrium of the re=sulting is a market equilibrium of the original market and vice versa.

Let $p = (p_1, p_2, \dots, p_m)$ be a price vector. Then for each agent a_i, $1 \le i \le n$ his utility per EURO of good j is

$$\frac{u_{ij}}{p_j}.$$

This quotient $\frac{u_{ij}}{p_j}$ is called the <u>relative utility</u> of good j for agent a_i.

For buying an optimal bundle of goods, each agent want to buy only goods which maximize his relative utility. Hence, we define for $1 \le i \le n$

$$\alpha_i := \max\left\{ \frac{u_{ij}}{p_j} \mid 1 \le j \le m \right\}.$$

Each agent $a_i \in B$ buys only goods j with $\frac{u_{ij}}{p_j} = \alpha_i$. For the identification of these goods we define the so-called <u>identification graph</u> $G(\bar{p}) := (A, B, E_{\bar{p}})$ where

$$E_{\bar{p}} := \left\{ (a_i, j) \mid \frac{u_{ij}}{p_j} = \alpha_i \right\}.$$

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Assume that the agents are indexed such that

$$e_i \geq e_{i+1} \text{ for } 1 \leq i \leq n.$$

For the computation of a market equilibrium we start with a price vector $\bar{p} = (p_1, p_2, \dots, p_m)$ such that

1) $\sum_{j=1}^{m} p_j \leq e_n$ and
2) for each good $j \in \{1, 2, \dots, m\}$ there is at least one agent $a_i \in B$ with

$$\frac{u_{ij}}{p_j} = \alpha_i$$

We can compute such a price vector $\bar{p}$ in the following way:

1. <u>for</u> $j := 1$ <u>until</u> m
   <u>do</u>
   $$p_j := \frac{e_n}{m}$$
   <u>od</u>;
2. <u>for</u> all j with $\frac{u_{ij}}{p_j} < \alpha_i$ for $1 \leq i \leq n$
   <u>do</u>
   $$p_j := \max \left\{ \frac{u_{ij}}{\alpha_i} \mid 1 \leq i \leq n \right\}$$
   <u>od</u>.

Note that in $G(\bar{p})$ each agent is connected with at least one good and also each good is connected with at least one agent.

Because of the first property of the price vector $\bar{p}$, each agent a_i can buy all j with $(a_i, j) \in E_{\bar{p}}$

Example 8.1

$n = 4$, $m = 5$

$e = (140, 100, 60, 20)$

The following matrix defines the utility functions of the agents:

$i \backslash j$	1	2	3	4	5
1	20	10	5	1	5
2	10	20	4	2	8
3	10	20	8	1	8
4	15	15	15	2	10

After the first step of the computation of the price vector $\bar{p}$, each price is equal to $\frac{e_4}{5} = \frac{20}{5} = 4$. This gives us the price vector

$$\bar{p} = (4, 4, 4, 4, 4).$$

Let z_{ij} denote the utility per EURO of the good j for the agent a_i; i.e.,

$$z_{ij} = \frac{u_{ij}}{p_j}$$

The following matrix contains the values

$$z_{ij}, \ 1 \le i \le 4, \ 1 \le j \le 5$$

with respect to the price vector $\bar{p} = (4, 4, 4, 4, 4)$:

$i \backslash j$	1	2	3	4	5
1	$\frac{20}{4}$	$\frac{10}{4}$	$\frac{5}{4}$	$\frac{1}{4}$	$\frac{5}{4}$
2	$\frac{10}{4}$	$\frac{20}{4}$	$\frac{4}{4}$	$\frac{2}{4}$	$\frac{8}{4}$
3	$\frac{10}{4}$	$\frac{20}{4}$	$\frac{8}{4}$	$\frac{1}{4}$	$\frac{8}{4}$
4	$\frac{15}{4}$	$\frac{15}{4}$	$\frac{15}{4}$	$\frac{2}{4}$	$\frac{10}{4}$

$\Rightarrow$

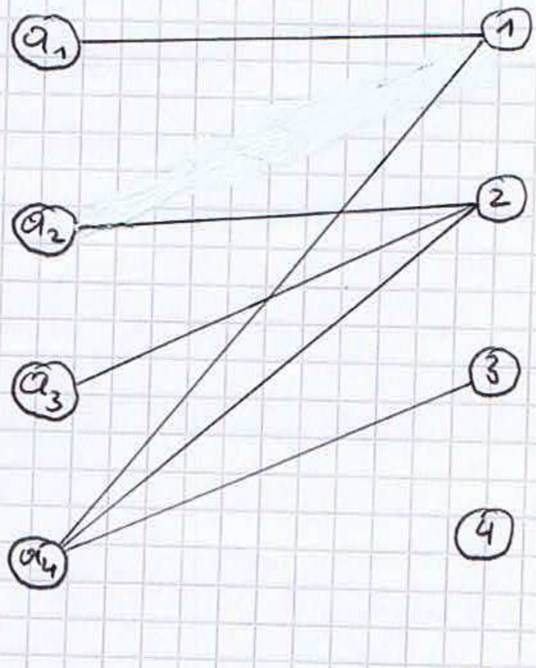
$$\alpha_1 = \frac{20}{4}$$

$$\alpha_2 = \frac{20}{4}$$

$$\alpha_3 = \frac{20}{4}$$

$$\alpha_4 = \frac{15}{4}$$

This gives us the following identification graph:



The goods 4 and 5 have no maximal relative utility for any agent. Hence, the prices p_4 and p_5 are updated in the second step of the computation of the price vector $\bar{p}$ in the following way:

$$p_4 := \max\left\{ \frac{u_{i4}}{\alpha_i} \;\middle|\; 1 \leq i \leq 4 \right\}$$

$$= \max\left\{ \frac{1\cdot 4}{20}, \frac{2\cdot 4}{20}, \frac{1\cdot 4}{20}, \frac{2\cdot 4}{15} \right\}$$

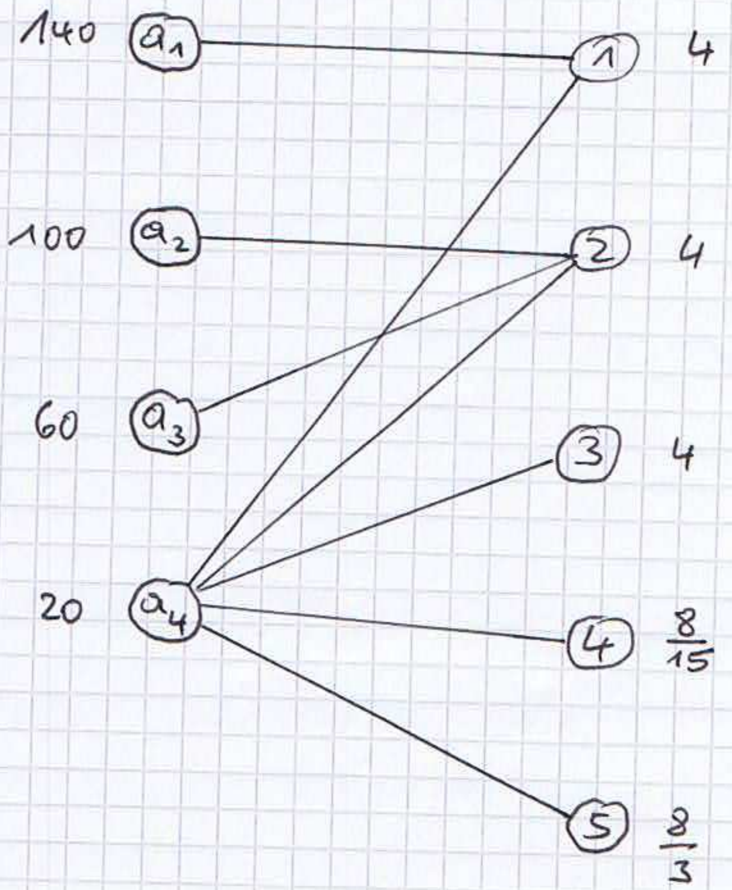
$$= \frac{8}{15}$$

$$p_5 := \max\left\{ \frac{u_{i5}}{\alpha_i} \;\middle|\; 1 \leq i \leq 4 \right\}$$

$$= \max\left\{ \frac{5\cdot 4}{20}, \frac{8\cdot 4}{20}, \frac{8\cdot 4}{20}, \frac{10\cdot 4}{15} \right\}$$

$$= \frac{40}{15} = \frac{8}{3}$$

This gives us the following identification graph where the prices and the endowments are written beside the nodes.



$\diamond$

The idea is the following.

We start with the price vector $\bar{p} = (p_1, p_2, \dots, p_m)$ and choose the agent a_1 which has the highest endowment.

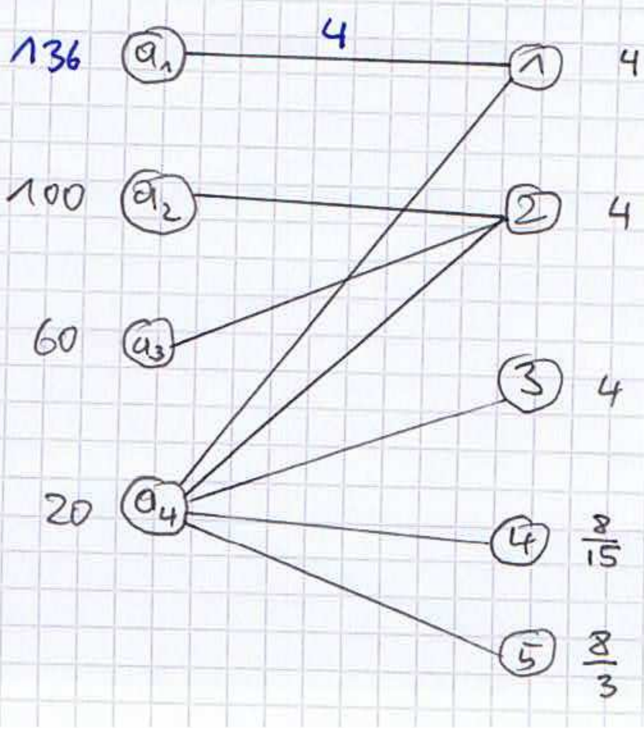
Since
$$\sum_{j=1}^{m} p_j \leq e_n \leq e_1 ,$$
the agent a_1 can buy all goods j with $(a_i, j) \in E_{\bar{p}}$.

We incorporate the situation in the identification graph by

- writing at the edge (a_i, j) the amount of money given for the good j by agent a_i and writing beside the agent a_i his free endowment.

Example 8.1 (continuation):

The identification graph at the moment that a_1 has buyed good 1 for a price of 4.



Our goal is to increase the prices of the goods connected with the agent a_1 such that agent a_1 uses a larger amount of money to buy these goods until the free endowment of this agent is zero.

The effect of increasing the prices of the goods connected with agent a_1 is the following.

- The relative utility of these goods decreases for all agents.

Example 8.1 (continuation)

The good 1 is the only good connected with the agent a_1. Furthermore,

$$\alpha_1 = \frac{u_{11}}{p_1} = \frac{20}{4} = 5$$

$$\frac{u_{12}}{p_2} = \frac{10}{4} = \frac{5}{2}$$

$$\frac{u_{13}}{p_3} = \frac{5}{4}$$

$$\frac{u_{14}}{p_4} = \frac{1 \cdot 15}{8} = \frac{15}{8}$$

$$\frac{u_{15}}{p_5} = \frac{5 \cdot 3}{8} = \frac{15}{8}$$

Let

$$\beta_1 := \max \left\{ \frac{u_{1j}}{p_j} \;\middle|\; 1 \leq j \leq m \text{ and } \frac{u_{1j}}{p_j} < \alpha_1 \right\}.$$

Note that $\beta_1 = \frac{5}{2}$.

Let p_1^* denote the increased price of good 1.

At the moment when

$$\frac{u_{11}}{p_1^*} = \beta_1 = \frac{u_{12}}{p_2}$$

we have to insert the edge $(a_1, 2)$ into the identification graph.

Note that

$$\frac{u_{11}}{p_1^*} = \frac{u_{12}}{p_2}$$

$$\Leftrightarrow p_1^* = \frac{u_{11}}{u_{12}} \cdot p_2$$

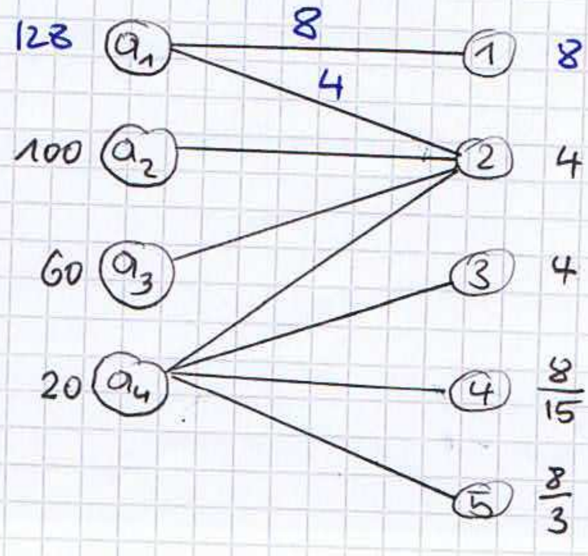
$$= 2 \cdot p_2$$

Hence, the moment arises for $p_1^* = 8$.

The agent a_1 buys the good 1 for a price of 8. and also the good 2 for a price of 4.

Because of the increase of the price of good 1, the edge $(a_4, 1)$ has to be deleted from the identification graph.

$\rightsquigarrow$



Let $p_1 = 8$. Now we increase the prices of the goods 1 and 2. Let p_1^* and p_2^* denote the increased prices. Note that

$$\alpha_1 = \frac{u_{11}}{p_1} = \frac{u_{12}}{p_2} = \frac{5}{2}.$$

All edges which describe the buying of a portion of a good has to remain an edge in the identification graph. Hence, after the increase of the prices of the goods 1 and 2 it has to be

$$\frac{u_{11}}{p_1^*} = \frac{u_{11}}{p_2^*}$$

$$\iff p_1^* = \frac{u_{12}}{u_{11}} p_2^* = 2 \cdot p_2^*$$

Note that

$$\beta_1 = \left\{ \frac{u_{1j}}{p_j} \;\middle|\; 1 \le j \le m \text{ and } \frac{u_{1j}}{p_j} > \alpha_1 \right\}$$

$$= \frac{u_{15}}{p_5} = \frac{u_{14}}{p_4} = \frac{15}{8}.$$

At the moment when

$$\frac{u_{11}}{p_1^*} = \frac{u_{12}}{p_2^*} = \frac{15}{8}$$

we have to insert the edges $(a_1, 4)$ and $(a_1, 5)$ into the identification graph. Note that

$$\frac{u_{12}}{p_2^*} = \frac{u_{15}}{p_5} \iff p_2^* = \frac{u_{12}}{u_{15}} p_5 = 2 \cdot p_5$$

Hence, $p_1^* = 2 \cdot p_2^* = 4 \cdot p_5$.

Hence, the moment arises for

$$p_1^* = 4 \cdot p_5 = 4 \cdot \frac{8}{3} = \frac{32}{3} \quad \text{and}$$

$$p_2^* = 2 \cdot p_5 = 2 \cdot \frac{8}{3} = \frac{16}{3}$$

The agent a_1 buys

- good 1 for a price of $\frac{32}{3}$,
- good 2 for a price of $\frac{16}{3}$,
- good 4 for a price of $\frac{8}{15}$, and
- good 5 for a price of $\frac{8}{3}$

Because of the increase of the price of good 2, the edge $(a_4, 2)$ has to be deleted from the identification graph. Furthermore, we have to check if the edge $(a_2, 2)$ and $(a_3, 2)$, resp. has to be replaced by another edge.

For doing this we have to compute the relative utilities of all goods for the agents a_2 and a_3 with respect to the price vector

$$p = (p_1, p_2, p_3, p_4, p_5) = \left(\frac{32}{3}, \frac{16}{3}, 4, \frac{8}{15}, \frac{8}{3}\right)$$

$\underline{a_2}$:

$$\frac{u_{21}}{p_1} = \frac{10 \cdot 3}{32} = \frac{30}{32} = \frac{15}{16} \qquad \frac{u_{25}}{p_5} = \frac{8 \cdot 3}{8} = 3$$

$$\frac{u_{22}}{p_2} = \frac{20 \cdot 3}{16} = \frac{60}{16} = \frac{30}{8} = \frac{15}{4}$$

$$\frac{u_{23}}{p_3} = \frac{4}{4} = 1$$

$$\frac{u_{24}}{p_4} = \frac{2 \cdot 15}{8} = \frac{30}{8} = \frac{15}{4} \implies \alpha_2 = \frac{15}{4}$$

Hence, $(a_2, 2)$ and $(a_2, 4)$ have to be edges in the identification graph.

$\underline{a_3}$:

$$\frac{u_{31}}{p_1} = \frac{10 \cdot 3}{32} = \frac{30}{32} = \frac{15}{16}$$

$$\frac{u_{32}}{p_2} = \frac{20 \cdot 3}{16} = \frac{15}{4}$$

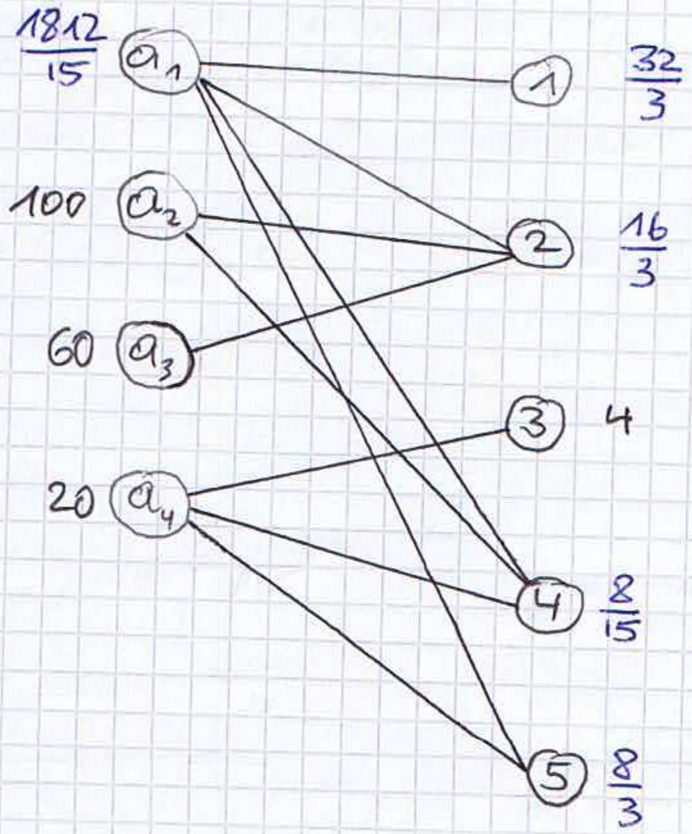
$$\frac{u_{33}}{p_3} = \frac{8}{4} = 2 \qquad \Rightarrow \alpha_3 = \frac{15}{4}$$

$$\frac{u_{34}}{p_4} = \frac{1 \cdot 15}{8} = \frac{15}{8}$$

$$\frac{u_{35}}{p_5} = \frac{8 \cdot 3}{8} = 3$$

Hence, $(a_3, 2)$ has to be an edge in the identification graph.

$\rightsquigarrow$



Now we increase the prices of the goods 1, 2, 4 and 5. Let p_1^*, p_2^*, p_4^* and p_5^* denote the increased prices. Note that

$$\alpha_1 = \frac{u_{11}}{p_1} = \frac{u_{12}}{p_2} = \frac{u_{14}}{p_4} = \frac{u_{15}}{p_5} = \frac{15}{8}$$

Since the edges $(a_1, 1)$, $(a_1, 2)$, $(a_1, 4)$ and $(a_1, 5)$ have to remain edges in the identification graph, it must be fulfilled:

$$p_1^* = 4 p_5^*$$
$$p_2^* = 2 p_5^*$$
$$p_4^* = \frac{u_{14}}{u_{15}} p_5^* = \frac{1}{5} p_5^*$$

Note that

$$\beta_1 = \frac{u_{13}}{p_3} = \frac{5}{4}$$

At the moment when

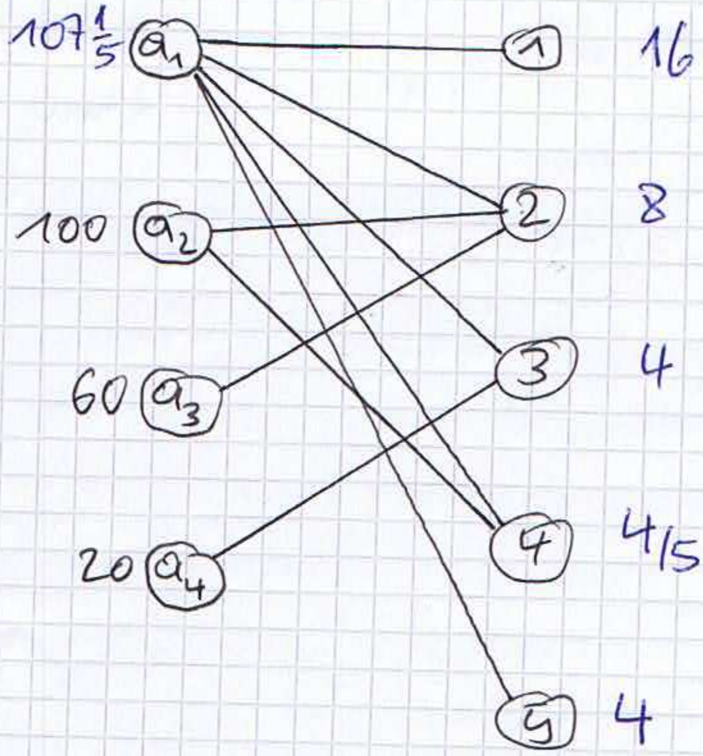
$$\frac{u_{15}}{p_5^*} = \frac{5}{4} \iff p_5^* = \frac{5 \cdot 4}{5} = 4$$

we have to insert the edge $(a_1, 3)$ into the identification graph. At this moment there holds

$$p_1^* = 4 p_5^* = 16$$
$$p_2^* = 2 p_5^* = 8$$
$$p_3 = 4$$
$$p_4^* = 1/5\, p_5^* = \frac{4}{5}$$
$$p_5^* = 4$$

Because of the increase of the prices of the goods 4 and 5 the edges $(a_4, 4)$ and $(a_4, 5)$ have to be deletet from the identification graph.

The good a_1 buys all goods.



Now we increase the prices of all goods until the agent a_1 has no additional money. Let p_1^*, p_2^*, p_3^*, p_4^* and p_5^* be the increased prices. Then there holds

$$\begin{aligned} p_1^* &= 4\,p_5^* \\ p_2^* &= 2\,p_5^* \\ p_3^* &= p_5^* \\ p_4^* &= 1/5\,p_5^* \end{aligned}$$

and

$$140 = p_1^* + p_2^* + p_3^* + p_4^* + p_5^*$$
$$= 4p_5^* + 2p_5^* + p_5^* + \frac{1}{5}p_5^* + p_5^*$$
$$= \frac{41}{5} p_5^*$$

$$\Leftrightarrow p_5^* = \frac{140 \cdot 5}{41} = \frac{700}{41}$$

Hence,

$$p_1^* = 4 \cdot p_5^* = \frac{2800}{41}$$
$$p_2^* = 2 \cdot p_5^* = \frac{1400}{41}$$
$$p_3^* = p_5^* = \frac{700}{41}$$
$$p_4^* = \frac{1}{5} p_5^* = \frac{140}{41}$$
$$p_5^* = \quad = \frac{700}{41}$$

Now, we choose the agent a_2. The goods 2 and 4 are the only good which are connected to agent 2 in the identification graph. First we compute the relative utilities of the goods for the agent 2.

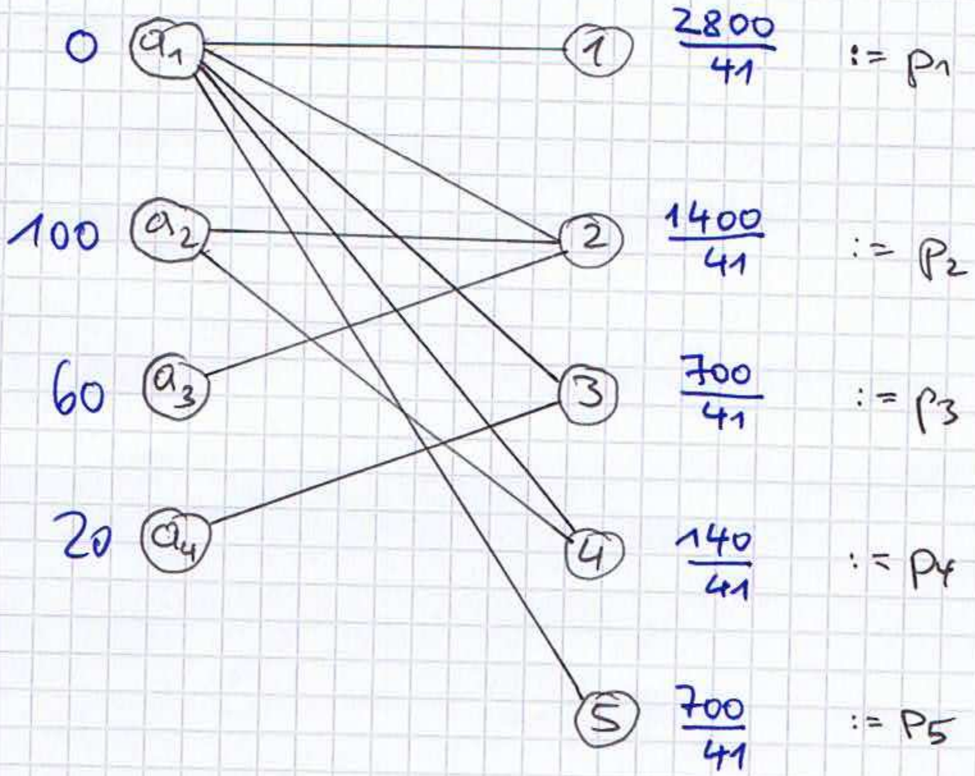
$$\frac{u_{21}}{p_1} = \frac{10 \cdot 41}{2800} = \frac{41}{280} \qquad \frac{u_{25}}{p_5} = \frac{8 \cdot 41}{700} = \frac{82}{175}$$

$$\frac{u_{22}}{p_2} = \frac{20 \cdot 41}{1400} = \frac{41}{70}$$

$$\frac{u_{23}}{p_3} = \frac{4 \cdot 41}{700} = \frac{41}{175} \qquad \Rightarrow \alpha_2 = \frac{41}{70}$$

$$\frac{u_{24}}{p_4} = \frac{2 \cdot 41}{140} = \frac{41}{70} \qquad \beta_2 = \frac{82}{175}$$

The current identification graph is the following



Our goal is to increase the prices of the goods connected to the agent a_2 until a_2 has no additional endowment.

At the moment, the goods 2 and 4 are connected to the agent a_2. Both goods are also connected to the agent a_1 which buys these goods. Agent a_1 is also connected to other goods. If we would only increase the prices of the goods 2 and 4, these goods would loss the property to be of maximal relative utility for the agent a_1 such that a_1 could not buy these goods. Hence, we have also to increase the prices of the other goods which are connected to a_1 such that the goods 2 and 4 remain to be of maximal relative utility for the agent a_1.

The price increase of the goods which are not connected to agent a_2 has to be payed by the agent a_1. Since a_1 has no additional money, he has to reduce the buyed quantity of the goods 2 and 4 so that he can pay the price increase of the other goods. This reduction can only be as large as the price which he pays for the goods 2 and 4. Then, the agent a_2 can buy this quantity of the goods 2 and 4 not buyed by the agent a_1.

Let X denote the total price increase of the goods 1, 2, 3, 4 and 5. This total price increase has to be payed by agent a_2 which has an endowment of 100. Hence,

$$X \leq 100.$$

How to distribute X to the goods 1, 2, 3, 4 and 5? Since all goods have to be connected to agent a_1 after the price increase, we have to distribute X in the same manner as $\tilde{p} := p_1 + p_2 + \ldots + p_5 = \frac{5740}{41}$ is distributed to the goods 1, 2, 3, 4 and 5. Note that

$$\frac{u_{1i}}{p_i} = \frac{u_{1j}}{p_j} \iff \frac{u_{1i}}{u_{1j}} = \frac{p_i}{p_j}.$$

This means that the ratios of the prices have to be the same as before. Hence,

$$p_i^* := p_i + \frac{p_i}{\tilde{p}} \cdot X \qquad \text{for } 1 \leq i \leq 5.$$

Note that

$$\frac{p_i^*}{p_j^*} = \frac{p_i + \frac{p_i}{\tilde{p}} X}{p_j + \frac{p_j}{\tilde{p}} X}$$

$$= \frac{\frac{p_i}{\tilde{p}}(\tilde{p} + X)}{\frac{p_j}{\tilde{p}}(\tilde{p} + X)} = \frac{p_i}{p_j}.$$

Furthermore,

$$\frac{p_1}{\tilde{p}} = \frac{2800}{5740} = \frac{20}{41} \qquad \frac{p_4}{\tilde{p}} = \frac{140}{5740} = \frac{1}{41}$$

$$\frac{p_2}{\tilde{p}} = \frac{1400}{5740} = \frac{10}{41} \qquad \frac{p_5}{\tilde{p}} = \frac{700}{5740} = \frac{5}{41}$$

$$\frac{p_3}{\tilde{p}} = \frac{700}{5740} = \frac{5}{41}$$

26.06.

As observed above, the price increase of the goods 1, 3 and 5 can be at most $p_2 + p_4 = \frac{1540}{41}$. Hence,

$$\left(\frac{20}{41} + \frac{5}{41} + \frac{5}{41}\right) \cdot X \leq \frac{1540}{41}$$

$$\Leftrightarrow \quad 30 \cdot X \leq 1540$$

$$\Leftrightarrow \quad X \leq 51\tfrac{1}{3}$$

Altogether, we obtain

$$X = 51\tfrac{1}{3} = \frac{154}{3}.$$

Hence,

$$p_1^* = p_1 + \frac{20}{41} \cdot \frac{154}{3} = \frac{3 \cdot 2800 + 20 \cdot 154}{41 \cdot 3}$$

$$= \frac{11.080}{123}$$

$$p_2^* = p_2 + \frac{10}{41}\cdot\frac{154}{3} = \frac{3\cdot 1400 + 10\cdot 154}{123}$$

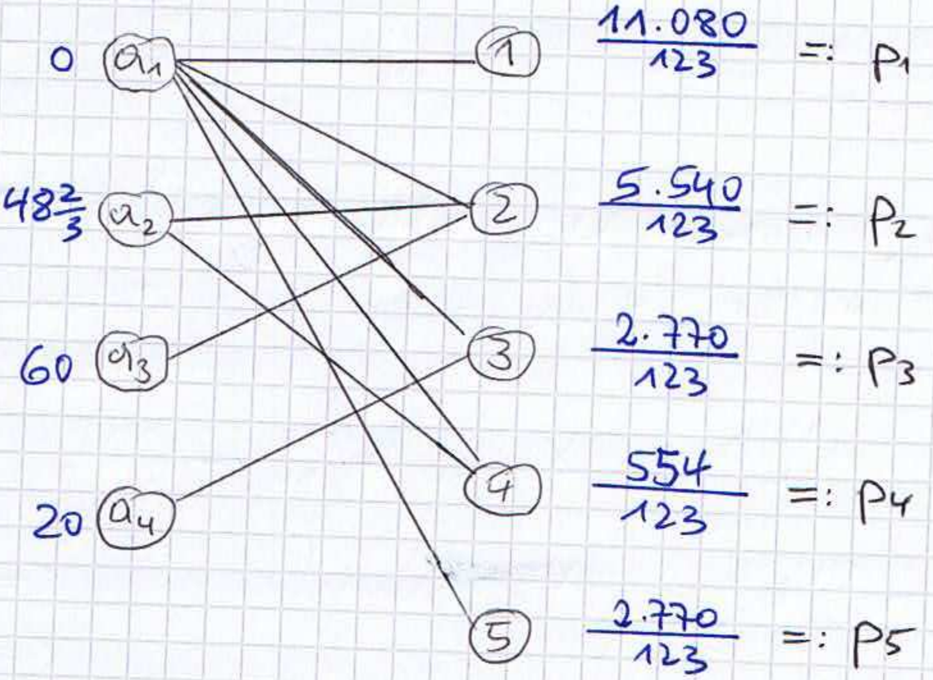
$$= \frac{5.540}{123}$$

$$p_3^* = p_3 + \frac{5}{41}\,\frac{154}{3} = \frac{2.770}{123}$$

$$p_4^* = p_4 + \frac{1}{41}\cdot\frac{154}{3} = \frac{554}{123}$$

$$p_5^* = p_5 + \frac{5}{41}\,\frac{154}{3} = \frac{2.770}{123}$$

Altogether, we obtain the following identification graph. Note that the agent a_1 is connected to the goods 2 and 4 but he buys nothing of these goods.



Now we increase the prices of the goods 2 and 4. After the increase of these prices these goods are not of maximal utility for the agent a_1. Hence, the edges $(a_1, 2)$ and $(a_1, 4)$ have to be deleted from the identification graph.

Next we compute the relative utilities of the goods for the agent a_2. (18)

$$\frac{u_{21}}{p_1} = \frac{10 \cdot 123}{11.080} = \frac{123}{1.108}$$

$$\frac{u_{22}}{p_2} = \frac{20 \cdot 123}{5.540} = \frac{123}{277} \implies \alpha_2 = \frac{123}{277}$$

$$\frac{u_{23}}{p_3} = \frac{4 \cdot 123}{2.770} = \frac{2 \cdot 123}{1.385} \qquad \beta_2 = \frac{4 \cdot 123}{1.385}$$

$$\frac{u_{24}}{p_4} = \frac{2 \cdot 123}{554} = \frac{123}{277}$$

$$\frac{u_{25}}{p_5} = \frac{8 \cdot 123}{2.770} = \frac{4 \cdot 123}{1.385}$$

Let

$$\tilde{p} := p_2 + p_4 = \frac{6.094}{123}$$

Then

$$\frac{p_2}{\tilde{p}} = \frac{5.540}{6.094} = \frac{2.770}{3.047}$$

$$\frac{p_4}{\tilde{p}} = \frac{554}{6.094} = \frac{277}{3.047}$$

As above, we obtain the following condition:

$$X \leq 48\tfrac{2}{3}.$$

Because of the increase of the prices of the goods 2 and 4, the relative utility of these goods for a_2 decreases. At the moment when the relative utility is β_2, we have to insert the edge $(a_2, 5)$. This occurs at the moment when

$$\frac{u_{24}}{p_2^*} = \frac{2}{\frac{554}{123} + \frac{277}{3.047}\cdot x} = \frac{4\cdot 123}{1.385}$$

$$\Leftrightarrow \frac{4\cdot 123}{1385}\left(\frac{554}{123} + \frac{277}{3.047}x\right) = 2$$

$$\Leftrightarrow \frac{277}{3.047}x = \frac{2\cdot 1385}{4\cdot 123} - \frac{554}{123}$$

$$\Leftrightarrow x = \left(\frac{1385 - 1108}{2\cdot 123}\right)\frac{3.047}{277}$$

$$\Leftrightarrow x = \frac{277\cdot 3047}{2\cdot 123\cdot 277}$$

$$= \frac{3047}{2\cdot 123}$$

$$= \frac{3047}{246} = 12\frac{95}{246}$$

Exercise:

Continue the example.

Goal: The description of the algorithm.

Assume that we have computed the initial price vector $\overline{p} = (p_1, p_2, \dots, p_m)$ such that

1) $\sum_{j=1}^{m} p_j \leq e_n$ and

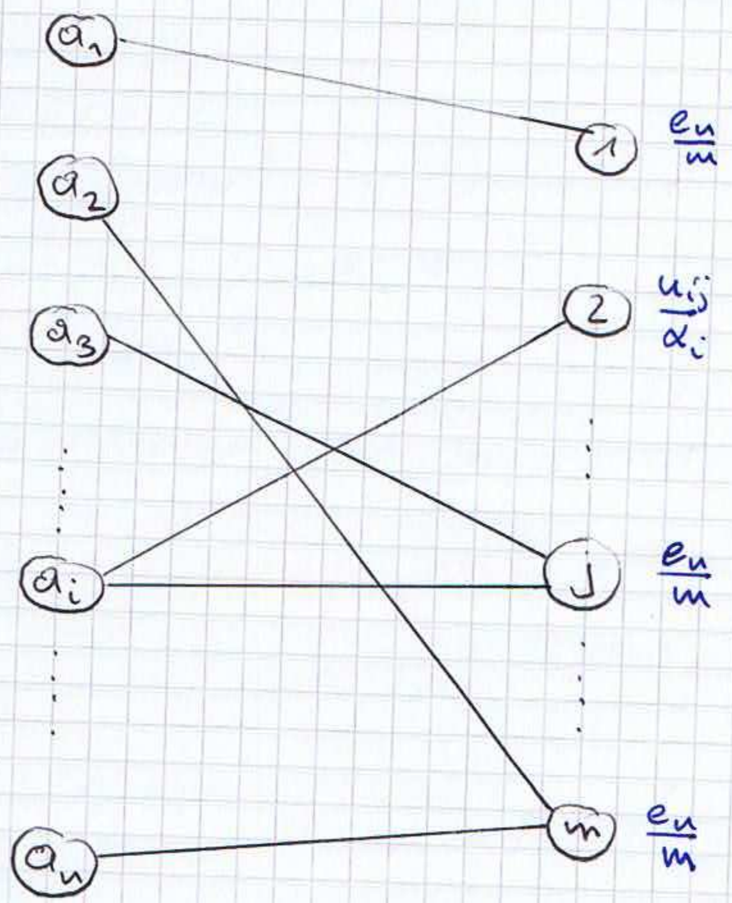
2) for each good $j \in \{1, 2, \dots, m\}$ there is at least one agent $a_i \in B$ with $\frac{u_{ij}}{p_j} = \alpha_i$

in the following way

1. **for** $j = 1$ **until** m
   do
   $$p_j = \frac{e_n}{m}$$
   od;

2. **for** all j with $\frac{u_{ij}}{p_j} < \alpha_i$ for $1 \leq i \leq n$
   do
   $$p_j := \max \left\{ \frac{u_{ij}}{\alpha_i} \;\middle|\; 1 \leq i \leq n \right\}$$
   od.

Let us consider the identification graph
$G(\bar{p}) = (A, B, E_p)$



Properties:

1) After the performance of Step 1 of the algorithm all goods have the same price $\frac{e_n}{m}$. Hence, in the identification graph each agent a_i is connected exactly to those goods which has maximal utility for him. Note that at least one good has this property.

2) Step 2 of the algorithm reduces the price of each good j which is not connected to any agent by Step 1 such that

   a) all edges introduced by Step 1 remain to be an edge in the identification graph and

   b) the good j is connected to at least one of the agents.

3) All goods are not sold out. This is trivial since no agent has buyed anything up to now.

We say that agent a_i is saturated if he has spend his whole money for buying goods.

The main loop of the algorithm is the following:

for $i := 1$ **step** 1 **until** n
  do
    SATURATE (a_i)
  od.

SATURATE(a_i) take care that the agent a_i spend his whole endowment for buying goods. Before describing the procedure SATURATE in detail we shall analyze the situations and the things which have to be performed during the first few calls of the procedure by the main loop of the algorithm.

a_1:

- At the beginning the agent a_1 has his total endowment e_1 and no portion of a good is buyed by any agent. Because of property 1 of the initial price vector $\bar{p} = (p_1, p_2, \ldots, p_m)$ agent a_1 can buy all goods which are connected to him.

$\rightsquigarrow$

(1) Agent a_1 buys all goods which are of maximal relative utility for a_1.

$\Rightarrow$

All these goods are sold out. Let $G(a_i)$ denote the set of goods having maximal relative utility for agent a_i.

If the agent a_1 has spend his whole endowment then a_1 is saturated.

Otherwise, we increase the prices of the goods in $G(a_1)$ until

i) another good becomes to be of maximal relative utility for a_1 or

ii) the total price increase X is equal to the part $r(e_1)$ of the endowment of agent a_1 which is not spend for buying goods.

We shall investigate both case one after the other.

i) Let β_1 denote the second best size of relative utility for agent a_1 before the price increase.

Because of the price increase, the maximal relative utility α_1 decreases. At the moment when $\alpha_1 = \beta_1$, we have to add the goods $j \in A \setminus G(a_1)$ with relative utility β_1 to (a_1).

Let $N(a_1)$ denote the set of these goods. and Let $P(N(a_1))$ be the sum of the prices of these goods.

Let X be the total price increase.