If the agent $a_1$ has spend his whole endowment then $a_1$ is saturated.

Otherwise, we increase the prices of the goods in $G(a_1)$ until

i) another good becomes to be of maximal relative utility for $a_1$ or

ii) the total price increase $X$ is equal to the part $r(e_1)$ of the endowment of agent $a_1$ which is not spend for buying goods.

We shall investigate both cases one after the other.

i) Let

$\cdot\ \beta_1 := \max\left\{ \frac{u_{1j}}{p_j} \;\middle|\; 1 \leq j \leq m \text{ and } \frac{u_{1j}}{p_j} < \alpha_1 \right\}$

denote the largest relative utility of a good for $a_1$ which is smaller then $\alpha_1$.

Let

$\cdot\ G^+(a_1) := \left\{ j \in A \setminus G(a_1) \;\middle|\; \frac{u_{1j}}{p_j} = \beta_1 \right\}.$

and

$\cdot\ P^+(a_1) := \sum_{j \in G^+(a_1)} p_j.$

Because of the price increase of the goods in $G(a_1)$, the relative utility $\alpha_1$ of these goods decreases. At the moment when

$\alpha_1 = \beta_1$, we have to add the goods in $G^+(a_1)$ to $G(a_1)$. This is performed in the following way:

(2) Agent $a_1$ buys all goods in $G(a_1)$ and pays the increased prices. Hence,

$r(e_1) := r(e_1) - X$

where $X$ is the total increase of the prices;

$G(a_1) := G(a_1) \cup G^+(a_1)$;

<u>if</u> $r(e_1) \geq P^+(a_1)$

<u>then</u>

$a_1$ buys all goods in $G^+(a_1)$;
$r(e_1) := r(e_1) - P^+(a_1)$;
<u>if</u> $r(e_1) = 0$
<u>then</u>
$a_1$ is saturated;
stop
<u>else</u>
increase the prices of the goods in $G(a_1)$
<u>fi</u>

<u>else</u>

```
while r(e_1) > 0
  do
    choose j ∈ G^+(a_1);
    G^+(a_1) := G^+(a_1) \ {j};
    if r(e_1) ≥ p_j
      then
        a_1 buys the good j;
        r(e_1) := r(e_1) - p_j
      else
        a_1 buys a portion r(e_1)/p_j
          of the good j;
        r(e_1) := 0
    fi
  od;
  a_1 is saturated;
  stop
fi.
```

**Properties:**

1) $G(a_1) = B_1 \cup B_2 \cup B_3$

where

- $j \in B_1 \Rightarrow j$ is sold out
- $|B_2| \leq 1$ and $j \in B_2 \Rightarrow$

  A portion of $j$ is buyed by $a_1$ but $j$ is <u>not</u> sold out

- $j \in B_3 \Rightarrow$ nothing of the good $j$ is buyed by the agent $a_1$.

2) $j$ not sold out $\Rightarrow$ $p_j$ is the initial price of the good $j$.

$\underline{a_2}$:

Let $p = (p_1, p_2, \dots, p_m)$ be the current price vector. Note

$$G(a_2) = \left\{ j \in B \mid \frac{u_{2j}}{p_j} = \alpha_2 \right\}$$

Let

$$C_1 := G(a_2) \cap B_1$$

$$C_2 := G(a_2) \cap B_2$$

$$C_3 := G(a_2) \setminus (C_1 \cup C_2)$$

Then

$$G(a_2) = C_1 \cup C_2 \cup C_3 .$$

Because of property 1 of the initial price vector and property 2 above, agent $a_2$ can buy all goods in $G(a_2)$ which are not sold out.

Construction $\Rightarrow$

These are exactly the goods in $C_2 \cup C_3$.

$\rightsquigarrow$

(1) Agent $a_2$ buys all goods in $C_2 \cup C_3$.

$\Rightarrow$

All these goods are sold out and hence, all goods in $G(a_2)$ are sold out.

If the agent $a_2$ has spend his whole endowment then $a_2$ is saturated.

28.06.

Otherwise, we increase the prices of all goods in $G(a_2)$ plus eventually the prices of some other goods. Let $r(e_2)$ be the free endowment of the agent $a_2$.

For the decision for which goods not in $G(a_2)$ the prices have to be increased we distinguish two cases.

Case 1: $G(a_2) \cap G(a_1) = \emptyset$

Then we increase the prices of the goods in $G(a_2)$ until

i) another good becomes to be of maximal relative utility for $a_2$ or

ii) the total price increase $X$ is equal to $r(e_2)$.

Note in the case that a good in $G(a_1)$ becomes to be of maximal relative utility for $a_2$ then $G(a_2) \cap G(a_1) \neq \emptyset$.

Case 2: $G(a_2) \cap G(a_1) \neq \emptyset$

If $a_1$ buys something in $G(a_2) \cap G(a_1)$ and $G(a_1) \nsubseteq G(a_2)$, an increase of the prices of the goods in $G(a_2)$ without an accurate increase of the prices of the goods in $G(a_1) \setminus G(a_2)$ would decrease the relative utility of the goods in $G(a_2) \cap G(a_1)$ but the maximal relative utility $\alpha_1$ would be the same as before.

$\Rightarrow$

The goods in $G(a_2) \cap G(a_1)$ would lose the property to be of maximal utility for agent $a_1$ such that $a_1$ cannot buy goods in $G(a_2) \cap G(a_1)$ as before.

$\Rightarrow$

An accurate price increase for all goods in $G(a_1)$ is necessary.

Before increasing prices of goods in $G(a_2) \cup G(a_1)$, we take care that all goods in $G(a_2) \cup G(a_1)$ are sold out.

Because of Step 1, all goods in $G(a_2)$ are sold out. But it is possible that some goods in $G(a_1) \setminus G(a_2)$ are not sold out. Since these goods are not of maximal relative utility for agent $a_2$, the agent $a_2$ cannot buy these goods directly.

$\rightsquigarrow$

Agent $a_2$ buys these goods indirectly as follows:

- $a_1$ buys fewer goods in $C_1 \cup C_2$ such that he can use the saved money for buying free goods (i.e., goods which are not sold out) in $G(a_1) \setminus G(a_2)$.

Let

$$a_1(C_2) := \begin{cases} \text{the price of the portion of the unique good } j \in C_2 \text{ buyed by } a_1 & \text{if } C_2 \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and let

$$p^-(C_1 \cup C_2) := a_1(C_2) + \sum_{j \in C_1} p_j .$$

Exactly the goods in $(B_2 \cup B_3) \setminus G(a_2)$ are the goods in $G(a_1) \setminus G(a_2)$ which are not sold out.

Let

$$a_1(B_2 \setminus G(a_2)) := \begin{cases} \text{the price of the portion of the unique good } j \in B_2 \setminus G(a_2) \text{ buyed by } a_1 & \text{if } B_2 \setminus G(a_2) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and let

$$p((B_2 \cup B_3) \setminus G(a_2)) := a_1(B_2 \setminus G(a_2)) + \sum_{j \in B_3 \setminus G(a_2)} p_j .$$

Obviously, at least one of $a_1(C_2)$ and $a_1(B_2 \setminus G(a_2))$ has to be zero.

Let

$$M := \min\{r(e_2), P^-(C_1 \cup C_2), P((B_2 \cup B_3) \setminus G(a_2))\}.$$

The following algorithm take care that at least one of the situations

i) $a_2$ is saturated,
ii) $a_1$ buys nothing in $G(a_2) \cap G(a_1)$, and
iii) all goods in $G(a_2) \cup G(a_1)$ are sold out

arises.

(2). $M' := M$ ;

- <u>if</u> $M' \geq a_1(C_2)$ <u>and</u> $r(e_2) \geq a_1(C_2)$
  <u>then</u>
  agent $a_2$ buys goods for the price of $a_1(C_2)$ in $C_2$;
  $r(e_2) := r(e_2) - a_1(C_2)$;
  $M' := M' - a_1(C_2)$;
  $C_2 := \emptyset$
  <u>else</u>
  agent $a_2$ buys goods for the price of $M'$ in $C_2$;
  $r(e_2) := r(e_2) - M'$;
  $M' := 0$

  fi

- **while** $M' > 0$
  **do**
    choose $j \in C_1$;
    $C_1 := C_1 \setminus \{j\}$;
    **if** $M' \geq p_j$
      **then**
        $a_2$ buys the good $j$;
        $r(e_2) := r(e_2) - p_j$;
        $M' := M' - p_j$
      **else**
        $a_2$ buys a portion of $\frac{M'}{p_j}$ of the good $j$;
        $M' := 0$
    **fi**
  **od**;
- **if** $M \geq P((B_2 \cup B_3) \setminus G(a_2))$
  **then**
    agent $a_1$ buys all goods in $(B_2 \cup B_3) \setminus G(a_2)$;
    (* all goods in $G(a_2) \cup G(a_1)$ are sold out *)
  **else**
    agent $a_1$ buys goods in $(B_2 \cup B_3) \setminus G(a_2)$ for the price of
    $$\min\{P^{-}(C_1 \cup C_2), P((B_2 \cup B_3) \setminus G(a_2))\}$$
    such that there is at most one good in $C_1 \cup C_2$ or in $(B_2 \cup B_3) \setminus G(a_2)$ with the property that $a_1$ buys only a portion of this good.

Exercise:
Prove that at least one of the situations i), ii) and iii) arises.

If situation
- i) arises then we are ready
- ii) arises then the prices of exactly the goods in $G(a_2)$ are increased. (Note that we are in Case 1.)
- iii) arises then the prices of the goods in $G(a_2) \cup G(a_1)$ are increased.

Assume that we increases the prices of the goods in $G(a_2) \cup G(a_1)$.

We have to investigate the effect of the price increase to the goods in $G(a_1)$. Note that

- agent $a_2$ can only buy goods in $G(a_2)$

$\Rightarrow$

The price increase of goods in $G(a_1) \setminus G(a_2)$ has to be payed by agent $a_1$.

Since $r(e_1) = 0$ this would only be possible if $a_1$ reduces the portion of goods in $C_1$ buyed by him such that he can pay the price increase of the goods in $G(a_1) \setminus G(a_2)$. Let

$$P^-(C_1) := \sum_{j \in C_1} p_j$$

where $a_1(j)$ is the price payed by agent $a_1$ for the portion of the good $j$ which he buys.

Then the price increase of the goods in $G(a_1) \setminus G(a_2)$ has to be

$$\leq P^-(C_1).$$

Let $X$ be the total price increase of the goods in $G(a_2) \cup G(a_1)$. We have to answer the following question:

In dependence upon $X$, what is the price increase $X'$ of the goods in $G(a_1) \setminus G(a_2)$?

Let

$$\tilde{p} := \sum_{j \in G(a_1) \cup G(a_2)} p_j.$$

Since all goods in $G(a_1) \cup G(a_2)$ have to remain in $G(a_1) \cup G(a_2)$ we distribute the price increase $X$ proportionally to the goods in $G(a_1) \cup G(a_2)$. Hence, we obtain for each good $j \in G(a_1) \cup G(a_2)$:

$$p_j^* := p_j + \frac{p_j}{\tilde{p}} \cdot X.$$

Therefore

$$X' = \sum_{j \in G(a_1) \setminus G(a_2)} \frac{p_j}{\tilde{p}} X$$

Since

$$X' \le p^-(C_1 \cup C_2)$$

we obtain

$$X \cdot \sum_{j \in G(a_1) \setminus G(a_2)} \frac{p_j}{\tilde{p}} \le p^-(C_1 \cup C_2)$$

$$\Longleftrightarrow \qquad X \le \frac{\tilde{p} \cdot p^-(C_1 \cup C_2)}{\sum_{j \in G(a_1) \setminus G(a_2)} p_j}$$

Beside of $X \le r(a_2)$ this gives us another upper bound for the size of $X$.

03.07.

We have also to compute a further upper bound which comes from the possibility that a good not in $G(a_1) \cup G(a_2)$ becomes to be of maximal relative utility for $a_2$ or for $a_1$. Note that because of the increase of the prices of the goods in $G(a_1) \cup G(a_2)$ the relative utilities of these goods decreases.

Let

$$\beta_1 := \max \left\{ \frac{u_{1j}}{p_j} \;\middle|\; j \in A \setminus (G(a_1) \cup G(a_2)) \right\}$$

$$G^+(a_1) := \left\{ j \in A \setminus (G(a_1) \cup G(a_2)) \;\middle|\; \frac{u_{1j}}{p_j} = \beta_1 \right\}$$

and