Therefore

$$X' = \sum_{j \in G(a_1) \setminus G(a_2)} \frac{p_j}{\tilde{p}} X$$

Since

$$X' \leq p^-(C_1 \cup C_2)$$

we obtain

$$X \cdot \sum_{j \in G(a_1) \setminus G(a_2)} \frac{p_j}{\tilde{p}} \leq p^-(C_1 \cup C_2)$$

$$\Leftrightarrow \quad X \leq \frac{\tilde{p} \cdot p^-(C_1 \cup C_2)}{\sum_{j \in G(a_1) \setminus G(a_2)} p_j}$$

Beside of $X \leq r(a_2)$ this gives us another upper bound for the size of X.

03.07.

We have also to compute a further upper bound which comes from the possibility that a good not in $G(a_1) \cup G(a_2)$ becomes to be of maximal relative utility for a_2 or for a_1. Note that because of the increase of the prices of the goods in $G(a_1) \cup G(a_2)$ the relative utilities of these goods decreases.

Let G_I be the set of goods which have to be increased

$$\beta_2 := \max \left\{ \frac{u_{2j}}{p_j} \,\middle|\, j \in A \setminus G_I \right\}$$

$$G^+(a_2) := \left\{ j \in A \setminus G_I \,\middle|\, \frac{u_{2j}}{p_j} = \beta_2 \right\}$$

Obviously, $G(a_2) \subseteq G_I$.

At the moment when $\alpha_2 = \beta_2$ we have to add the goods in $G^+(a_2)$ to $G(a_2)$. Hence, we have to answer the following question:

What is the total price increase which leads to $\alpha_2 = \beta_2$?

Let $j' \in G(a_2)$; i.e., $\frac{u_{2j'}}{p_{j'}} = \alpha_2$ and let

$$\tilde{p} := \sum_{j \in G_I} p_j$$

We have to compute the total price increase X such that

$$\frac{u_{2j'}}{p_{j'} + \frac{p_{j'}}{\tilde{p}} X} = \beta_2$$

$$\Longleftrightarrow \quad \frac{u_{2j'}}{\beta_2} = p_{j'} + \frac{p_{j'}}{\tilde{p}} X$$

$$\Longleftrightarrow \quad X = \left(\frac{\alpha_2}{\beta_2} - 1\right) \tilde{p}$$

This means that $\left(\frac{\alpha_2}{\beta_2} - 1\right) \tilde{p}$ is the total price increase which leads to $\alpha_2 = \beta_2$.

If $G(a_1) \subseteq G_I$ then we have to answer the analogous question with respect to α_1 and β_2. Then we obtain

$$X = \left(\frac{\alpha_1}{\beta_1} - 1\right) \tilde{p}.$$

Hence, the third upper bound is

$$\min \left\{ \left(\frac{\alpha_e}{\beta_e} - 1\right) \tilde{p} \;\middle|\; G(a_e) \subseteq G_I \right\}.$$

Altogether, we choose the total price increase

$$X := \min \left\{ r(a_2), \frac{P^-(C_1 \cup C_2)}{\sum_{j \in G(a_1) \setminus G(a_2)} p_j} \cdot \tilde{p}, \min \left\{ \left(\frac{\alpha_e}{\beta_e} - 1\right) \tilde{p} \mid G(a_e) \subseteq G_{\underline{I}} \right\} \right.$$

In contrast to the saturation of the agent a_1 some further situations arise. Hence, we have to modify and to extend the algorithm for the saturation of a_1 to obtain an algorithm for the saturation of a_2. Instead of doing this, we consider the general case directly; i.e., we describe SATURATE(a_i), $i \geq 1$.

<u>SATURATE(a_i)</u>:

Let $\bar{p} = (p_1, p_2, \dots, p_m)$ be the current price vector and $\bar{p}^* = (p_1^*, p_2^*, \dots, p_m^*)$ be the price vector after the price increase. Let

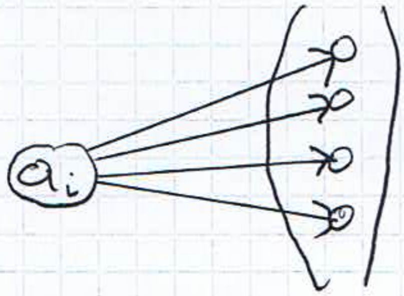
$$\alpha_i := \max \left\{ \frac{u_{ij}}{p_j} \mid 1 \leq j \leq m \right\} \quad \text{and}$$

$$G(a_i) := \left\{ j \in A \mid \frac{u_{ij}}{p_j} = \alpha_i \right\}.$$

$G(a_i)$ contains exactly the goods which can be buyed by agent a_i. Possibly, some goods in $G(a_i)$ are not buyed by agent a_i. Let

$$G'(a_i) := \{ j \in G(a_i) \mid a_i \text{ buys something of } j \}$$

We can express this by the following bipartite graph:



$G(a_i)$

We write at the edge (a_i, j') the amount $m(a_i, j')$ of money payed by a_i for the good j'.

Note that at the begining, a_i buys nothing such that $m(a_i, j') = 0 \quad \forall j' \in G(a_i)$.

The algorithm will maintain the following invariant:

<u>I1</u>:

a) If the price of a good j is increased then the good j is sold out.

b) A good j which is sold out remains to be sold out during the whole algorithm.

$\Rightarrow$

Each good in $G(a_i)$ which is not sold out has still his initial price.

$\Rightarrow$

a_i can buy all these goods such that after doing this all goods in $G(a_i)$ are sold out.

Let $r(e_i)$ be the endowment of agent a_i after buying these goods.

<u>Goal</u>:

Reduction of $r(e_i)$ by increasing the prices of the goods in $G(a_i)$.

Consider $j \in G(a_i)$. If there is an agent a_p such that a_p buys something of the good j then we have to increase also the prices of all other goods in $G(a_p)$. Hence we extend the bipartite graph above to a layered network as follows:

Let $L(a_i)$ be the set of all goods such that their prices have to be increased because of the price in= crease of the goods in $G(a_i)$. Let $S(a_i)$ be the set of agents a_p such that the prices of the goods in $G(a_p)$ have to be increased because of the price increase of the goods in $G(a_i)$

The layered network $\mathcal{G}(\bar{p}, a_i) = (S(a_i) \cup L(a_i), E(a_i$ can be computed as follows.

The layers of $\mathcal{G}(\bar{p}, a_i)$ contain alternately nodes in $S(a_i)$ and nodes in $L(a_i)$. Let

- S_ℓ, $\ell \geq 1$ be the ℓ-th layer with nodes in $S(a_i)$ and
- L_ℓ, $\ell \geq 1$ be the ℓ-th layer with nodes in $L(a_i)$.

Then we obtain

$$S_1 := \{a_i\}$$
$$L_1 := \{ j \in A \mid j \in G(a_i) \}$$
$$E_{11} := \{ (a_i, j) \mid j \in L_1 \}$$

For $\ell \geq 1$ we define

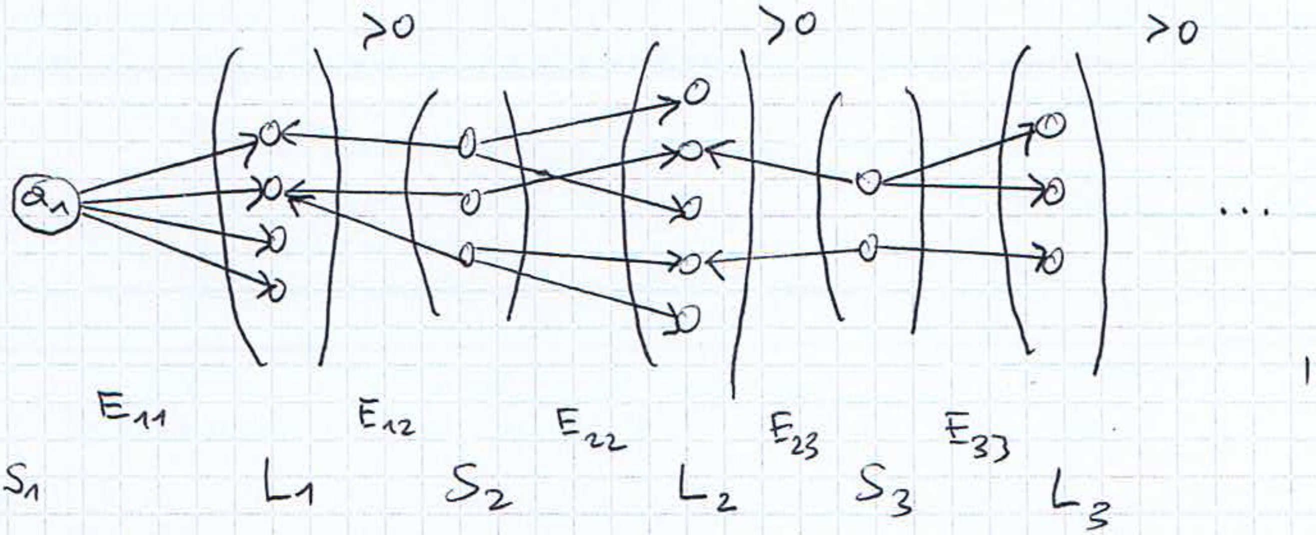
$$S_{\ell+1} := \{ a_{i'} \in B \setminus \bigcup_{h=1}^{\ell} S_h \mid \exists j \in L_\ell : j \in G'(a_{i'}) \}$$

$$E_{\ell\ell+1} := \{ (a_{i'}, j) \mid a_{i'} \in S_{\ell+1}, j \in L_\ell \text{ and } j \in G'(a_{i'}) \}$$

$$L_{\ell+1} := \{ j \in A \setminus \bigcup_{h=1}^{\ell} L_h \mid \exists a_{i'} \in S_{\ell+1} : j \in G(a_{i'}) \}$$

$$E_{\ell+1\ell+1} := \{ (a_{i'}, j') \mid a_{i'} \in S_{\ell+1}, j \in L_{\ell+1} \text{ and } j \in G(a_{i'}) \}$$

The following figure describes the structure of $\mathcal{G}(\bar{p}, a_i)$



Construction $\Rightarrow$

$$S(a_i) = \bigcup_\ell S_\ell, \quad L(a_i) = \bigcup_\ell L_\ell, \quad E(a_i) = \bigcup_\ell (E_{\ell\ell} \cup E_{\ell\ell+1})$$

We increase the prices of the goods in $L(a_i)$ (getting the new price vector $\bar{p}^* = (p_1^*, p_2^*, \dots, p_m^*)$) until the layered network changes his structure or $r(e_i)$ becomes zero.

$\mathcal{G}(\bar{p}, a_i)$ changes his structure if

i) an edge $(a_{i'}, j)$ leaves $E(a_i)$; i.e., $(a_{i'}, j)$ is an edge between layers L_ℓ and $S_{\ell+1}$, $\ell \geq 1$ and j leaves $G'(a_{i'})$ or

ii) a good j enters $L(a_i)$; i.e., there is an agent $a_{i'} \in S(a_i)$ such that the relative utility of the good j becomes to be maximal for $a_{i'}$.

Hence, we have to compute the minimal price increase M_1 such that an edge leaves the layered network $\mathcal{G}(\bar{p}, a_i)$ and also the minimal price increase M_2 such that a new good enters the layered network $\mathcal{G}(\bar{p}, a_i)$. Then

$$M := \min \{ M_1, M_2, r(e_i) \}$$

is the total price increase chosen for the current round.

05.07.

<u>The computation of M_1</u>:

The agent a_i has to pay the total price increase. Hence, for each good $j \in L(a_i)$ his price increase has to be transported from j to a_i through $\mathcal{G}(\bar{p}, a_i)$.

Let L_r be the last layer which contains goods. The price increase of the goods in L_r has to be payed by the agents in layer S_r. Since the free endowment of these agents is zero, they can only reduce the amount of money used for