Algorithmic Game Theory and the Internet

References

A: Game Theory

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B: Linear Programming

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0. Motivation

Reference:

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Since more than sixty years game theory is estab= lished as a field in economics. But the interest of the computer science community for algorithmic problems of the game theory arises only fifteen years ago. The reason for this interest comes from the growing important role of the Internet. Many problems which occur during handling the Internet can be modeled using game theory. The algorithmic solution of such problems implies often also a solution of a general al= gorithmic problem of the game theory.

Esspecially, we are interested in routing prob- lems of the Internet. Internet users act selfishly and spontaneously. There is no authority that monitors and regulates network operation to obtain some "social optimum" (e.g. minimum total delay. Hence, the following question arises:

How much performance is lost because of the lack of coordination?

The model

Given

- a network $G = (V, E)$,
- n users (which do not cooperate) and
- for each user B_i, $1 \le i \le n$
  - a source s_i,
  - a sink t_i and
  - an amount of traffic w_i which is to send from s_i to t_i.

P_i, $1 \le i \le n$ denotes the set of all simple paths from s_i to t_i in the network G. If every user has to send its whole traffic over the same path a strategy of user B_i is a path $P \in P_i$.

Question

What is a best or a useful strategy for the user B_i?

Note that similiar situations arise during the solution of economical and sociological problems. For the solution of these problems the game theory has been developed.

$\rightsquigarrow$

The game theory is of interest for the solution of Internet problems.

The game theory studies the interactions of agents (or players) and the strategic scenaries which arise during the solution of economical or sociological problems. The algorithmic game theory deals with computational problems which arise in game theory.

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1. Introduction in the game theory

First we shall define some basic notations of the game theory and explain these notations using some simple examples. Then we shall consider some special kinds of games which are important for routing problems in networks.

1.1 Strategic games

A binary relation $\gtrsim$ on a set A is complete if $\forall\, a, b \in A$ there holds $a \gtrsim b$ or $b \gtrsim a$.
A binary relation is called preference relation if the relation is complete, reflexive and transitive.

A strategic game $\Gamma = \langle N, (A_i), (\gtrsim_i) \rangle$ consists of

1. a finite set $N = \{1, 2, \dots, n\}$ of players,
2. for each player $i \in N$ a nonempty set A_i of actions (or strategies) which are possible for the player i, and
3. for each player $i \in N$ a preference relation $\gtrsim_i$ on $A := A_1 \times A_2 \times \dots \times A_n$.

The game Γ is finite if A_i is finite $\forall\, i \in N$.
A is the set action profiles.

A usual way to specify the preferences of the players is by assigning, for each player, a value to each outcome. I.e., for each player $i \in N$ we define a <u>utility function</u>

$$u_i : A \to \mathbb{R}$$

Such that

$$u_i(a) \geq u_i(b) \Leftarrow a \succsim_i b \quad \forall a, b \in A.$$

$u_i(a)$ denotes the <u>utility</u> of player i with respect to the action profile $a \in A$. If the preference relations $\succsim_i$, $i \in N$ are specified by utility functions $u_i : A \to \mathbb{R}$, $i \in N$ then we write also $\langle N, (A_i), (u_i) \rangle$ instead of $\langle N, (A_i), (\succsim_i) \rangle$.

<u>Exercise</u>:

Assume that A_i is countable for each player $i \in N$. Show that for every preference relation $\succsim_i$, $i \in N$ there is a utility function $u_i : A \to \mathbb{R}$ such that

$$u_i(a) \geq u_i(b) \Leftarrow a \succsim_i b \quad \forall a, b \in A.$$

Which properties must have the utility function u_i such that the other direction is also valid?

Given an action profile $a^* \in A$ each player $i \in N$ can change this profile by replacing its action

a_i^* in a^* by another action $a_i \in A_i \setminus \{a_i^*\}$. (a_{-i}^*, a_i) denotes the action profile which we obtain from a^* replacing a_i^* by a_i. I. e.,

$$(a_{-i}^*, a_i) = (a_1^*, a_2^*, \dots, a_{i-1}^*, a_i, a_{i+1}^*, \dots, a_n^*).$$

The goal of each player is to maximize its utility. This means that the player $i \in N$ would always replace his action in the current action profile a^* if this would increase his utility strictly. The state of a game is unstable if at least one player can improve its utility by the replacement of its action in the current action profile. Otherwise, the state of the game is stable. If the state of a game is stable then we say that the game is in a Nash equilibrium. More formally:

A <u>Nash equilibrium</u> of a strategic game $\Gamma = \langle N, (A_i), (\succsim_i) \rangle$ is an action profile $a^* \in A$ with the property that for all player $i \in N$ there holds

$$(a_{-i}^*, a_i^*) \succsim_i (a_{-i}^*, a_i) \quad \forall a_i \in A_i.$$

<u>Examples</u>:

a) <u>Bach or Stravinsky</u> (BoS)

At the same time there are two concerts, one with music of Johann Sebastian Bach and the other with music of Igor Stravinsky. Two players wish to visit together one of these two concerts. For both players it is most important to visit one of the concerts together, but one of the players prefers Bach and the other prefers Stravinsky.

We specify the preference relations using a payoff matrix.

	B	S
B	2,1	0,0
S	0,0	1,2

i-th number $\hat{=}$ utility of player i

BoS has two Nash equilibria: (B,B) and (S,S).

b) Coordination game

As in the BoS game two players wish to visit together one of two consERTS. But in contrast to BoS both players prefer the same concert. Hence, we obtain the following payoff matrix:

	C	L
C	2,2	0,0
L	0,0	1,1

Nash equilibria:

Chopin / Chopin and Liszt / Liszt.

In contrast to BoS both players have the interest to obtain the same Nash equilibrium Chopin/Chopin.

c) Head or taie

Each of two players chooses head or taie. If both players choose the same then player 2 obtains one Euro from player 1. Otherwise, player 1 obtains one Euro from player 2. Therefore, we obtain the following payoff matrix:

	H	T
H	-1, 1	1, -1
T	1, -1	-1, 1

no Nash equilibrium

For each action profile, the sum of the utilities is zero. Such a game is called zero-sum-game.

d) Prisoner's dilemma

Two prisoners are in different cells. If both confess to the crime then both obtain a prison sentence of three years. If one of the prisoners confess to the crime but the other remains silent then the chief witness obtains only one year and the other five years stay in the prison. If both remain to be silent then both will obtain two years for minor offenses. This leads to the following payoff matrix:

1st p. \ 2nd p.	C	S
C	-3, -3	-1, -5
S	-5, -1	-2, -2

Nash equilibrium:

C/C

Independently of the action of the other prisoner for each prisoner is the best action to confess to the crime.

In a strategic game each player tries selfishly to maximize its utility.

Given an action profile a^*, the <u>social value</u> of a^* is the sum of the utilities of all players. The <u>fairness</u> of a^* is the minimum utility of all players.

We shall analyze games which are motivated by applications in computer sciences. Besides the development of algorithms we shall investigate the following questions with respect to a given strategic game Γ:

- Has Γ a Nash equilibrium? Is this Nash equilibrium unique?
- How compares the social value of a Nash equilibrium with the best possible social value?

1.2 Load balancing games

The following simple load balancing game is well known in scheduling theory.

Given:

- m servers $s_1, s_2, \ldots, s_m$ which organizes web pages.
- n tasks $t_1, t_2, \ldots, t_n$ which request web pages,

  for each task t_j, $1 \leq j \leq n$

  - a size $p_j := |t_j|$
  - a subset $S_j \subseteq S := \{s_1, s_2, \ldots, s_m\}$ of servers which can reply the request t_j.

  Every task is selfish.

One seeks for an assignment

$$A : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$$

of the tasks to the servers.

Given an assignment A, the <u>load</u> l_i of the server s_i, $1 \le i \le m$ is the sum of the sizes of those tasks which are assigned to s_i by A. I.e.,

$$l_i := \sum_{j : A(j) = i} p_j .$$

<u>Remark</u>:

- As larger the load of a server as smaller its speed. Hence, each task looks for a server with a load as small as possible.

- <u>Nash equilibrium</u>:

  a) All servers are identical.

  We have a Nash equilibrium if

  $$A(j) = i \Rightarrow l_i \le l_k + p_j \quad \forall k \in S_j$$

  for all tasks $j \in \{t_1, t_2, \dots, t_n\}$.

  b) The servers are nonidentical.

  Assume that each server s_i has a load dependent responce time

  $$r_i : \mathbb{Q} \to \mathbb{Q}$$

  which is nonnegative and monotone increasi

Then we have a Nash equilibrium if

$$A(j) = i \Rightarrow r_i(\ell_i) \leq r_k(\ell_k + p_j) \quad \forall k \in S_j$$

for all tasks $j \in \{1, 2, \dots, n\}$.

Question:
Has each load balancing game a Nash equilibrium?

The next theorem gives an answer to this question for the case that all functions r_i, $1 \leq i \leq m$ are strongly monotone increasing.

Theorem 1.1
Assume that $r_i : \mathbb{Q} \to \mathbb{Q}$ are nonnegative and strongly monotone increasing for all servers s_i, $1 \leq i \leq m$. Then there is a Nash equilibrium for the load balancing game.

Proof:

Start with an arbitrary valid assignment

$$A : \{1, 2, \dots, n\} \to \{1, 2, \dots, m\}.$$

As long as possible apply the following operation:

- If there is a task j which can improve its response time by a valid change of its server then perform this server change.

Goal: Proof that the algorithm terminates.

Idea:

Definition of a function f where its value only depends on the current assignment such that the value of the function improves strictly after each operation.

$\Rightarrow$

No assignment repeats.

Since the number of different assignments is finite, the algorithm has to terminate.

Realization:

Let A be a valid assignment. Then we define

$$f(A) := r_{i_1}(l_{i_1}), r_{i_2}(l_{i_2}), \dots, r_{i_m}(l_{i_m})$$

such that

$$r_{i_j}(l_{i_j}) \geq r_{i_{(j+1)}}(l_{i_{(j+1)}}) \quad \text{for } 1 \leq j < m$$

This means that f(A) is the monotone decreasing sequence of response times of the servers.

Let A be the assignment before the performance of the current operation and A' be the assignment after the performance of the current operation.

Assume that the task t_j changes its server from s_i to s_k.

Let

$$p := |\{e \mid r_e(\ell_e) \geq r_i(\ell_i)\}|.$$

$\Rightarrow$

1) The response time $r_k(\ell_k)$ is in the sequence $f(A)$ to the right of the pth response time.

2) $r_i(\ell_i) > r_k(\ell_k + p_j)$

Next we investigate to the effect of the server exchange.

a) For all $e \in \{1, 2, \ldots, m\} \setminus \{i, k\}$ the response time of the server s_e does not change.

b) For $f(A')$, the following hold:

- The first $(p-1)$ response times remain unchanged.

- The p-th response time is
  - either $r_i(\ell_i - p_j)$,
  - or $r_k(\ell_k + p_j)$, load before server exchange
  - or $(p+1)$-th response time in $f(A)$.

In each case, the p-th response time has decreased strictly.

$\Rightarrow$

In the lexicographical order, the sequence of response times decreases strictly.

■

Remark:

The proof above contains an algorithm for the computation of a Nash equilibrium. But this algorithm can consider an exponential number of distinct assignments. Hence, its runtime can be exponential.

Exercise:

Is Theorem 1.1 also valid if we replace "strongly monotone increasing" by "monotone increasing". Prove your answer.

Goal:

Comparision of the maximal response time of a Nash equilibrium with the minimal possible maximal response time.

For an assignment A let

$$C(A) := \max \{ r_i(l_i) \mid 1 \leq i \leq m \}.$$

denote the maximal response time with respect to the assignment A.

Theorem 1.2

Assume that $r_i : \mathbb{Q} \to \mathbb{Q}$ are nonnegative and strongly monotone increasing for all servers s_i, $1 \leq i \leq m$. Then there is a Nash equilibrium A such that $C(A)$ is minimum with respect to all valid assignments.

Proof:

Let A' be an arbitrary valid assignment such that $C(A')$ is minimum.

Apply to A' the algorithm of the proof of Theorem 1.1. Note that this algorithm never increases the maximal response time of the current assignment.

Let A be the final assignment of the algorithm.

$\Longrightarrow$

$$C(A) \leq C(A')$$

■

<u>Question:</u>
Given an arbitrary Nash equilibrium. How large is its maximal response time in comparision with the best possible maximal response time?

We shall give an answer to this question for the case

- $S_j = \{1, 2, \dots, m\}$ for $1 \leq j \leq m$ and
- $r_i(x) = x$ for $1 \leq i \leq m$.

<u>Theorem 1.3</u>

Let Γ be a load balancing game with $r_i(x) = x$ $\forall$ server s_i, $1 \leq i \leq m$ and $S_j = S$ for all tasks t_j, $1 \leq j \leq n$. Let A' be an valid assignment with minimum maximal response time. Let A be an arbitrary Nash equilibrium. Then $C(A) \leq 2 \cdot C(A')$.

<u>Proof:</u>

Let

s_i be a server with $r_i(l_i) = C(M)$

and let

t_j be a task with $A(j) = i$.

A Nash equilibrium $\Rightarrow \ell_k + p_j \geq \ell_i \quad \forall k \in S$

If we sum over all servers then we obtain

$$\sum_{k=1}^{m} \ell_k \geq m(\ell_i - p_j)$$

$\Leftrightarrow$

$$C(M) = \ell_i \leq \frac{1}{m}\left(\sum_{k=1}^{m} \ell_k\right) + p_j.$$

<u>Goal</u>:

Proof that

- $\frac{1}{m} \sum_{k=1}^{m} \ell_k \leq C(A')$ and
- $p_j \leq C(A')$.

Since A' assigns the task t_j to a server it follows

$$C(A') \geq p_j.$$

Note that for all assignments there holds

$$\sum_{k=1}^{m} \ell_k = \sum_{j=1}^{n} p_j.$$

Hence, $C(A') \geq \frac{1}{m} \sum_{j=1}^{n} p_j$ implies

$$C(A') \geq \frac{1}{m} \sum_{k=1}^{m} \ell_k.$$

Goal:
The maximization of the social value of a solution; i.e., the minimization of the sum of the responsetimes.

We consider the case that all tasks have the same size; i.e., $p_j = 1$ for $1 \leq j \leq n$.

Let A be a valid assignment. The social value $S(A)$ is given by

$$S(A) = \sum_{i=1}^{m} \ell_i \cdot r_i(\ell_i)$$

Since the size of each task is one, the number of tasks on a server is equal its load. All tasks on server s_i have response time $r_i(\ell_i)$

Goal:
The definition of a function which allows an exact description of the change of the response time of a task t_j after a server exchange.

For a valid assignment A we define

$$\Phi(A) := \sum_{q=1}^{m} \sum_{\ell=1}^{\ell_q} r_q(\ell)$$

where ℓ_q is the load of server s_q with respect to the assignment A.

ϕ is called potential function.

Theorem 1.4

Let A be a valid assignment of a load balancing game Γ. Let A' be the assignment which is obtained from A by changing the server of task t_j from s_i to s_k. Then the change of the value of the potential function is equal to the change of the response time of the task t_j. I. e.,

$$\phi(A) - \phi(A') = r_i(l_i) - r_k(l_k+1).$$

Proof:

$$\phi(A)-\phi(A') = \sum_{q=1}^{m}\sum_{l=1}^{l_q} r_q(l) - \sum_{q=1}^{m}\sum_{l=1}^{l'_q} r_q(l)$$

where l'_q is the load of server s_q with respect to the assignment A'. Note that

$$l'_q = \begin{cases} l_q & \text{if } q \in \{1,2,\dots,m\} \setminus \{i,k\} \\ l_q - 1 & \text{if } q = i \\ l_q + 1 & \text{if } q = k \end{cases}$$

Hence, we obtain

$$\phi(A) - \phi(A') =$$

$$\left(\sum_{\ell=1}^{\ell_i} r_i(\ell) + \sum_{\ell=1}^{\ell_k} r_k(\ell)\right) - \left(\sum_{\ell=1}^{\ell_i-1} r_i(\ell) + \sum_{\ell=1}^{\ell_k+1} r_k(\ell)\right)$$

$$= r_i(\ell_i) - r_k(\ell_k+1)$$

∎

Remark:

The server exchange of task t_j changes the loads of the servers s_i and s_k. Hence, also the response times of the other tasks on the servers s_i and s_k are changed. The values of the potential function describes exactly the change of the response time of that task which has changed its server.

A game is called potential game if there is a potential function ϕ which has the following property:

- If a player i changes its strategy from s to s' and all other players do not change their strategy then the change of the value of the potential function ϕ is equal to the change of the utility of player i.

Hence, the special case of the load balancing game described above is a potential game.

Theorem 1.5

All potential games on finite sets of strategies have a Nash equilibrium. Especially, a solution such that the potential function ϕ has the minimum value is a Nash equilibrium.

Proof:

If the solution Z of a potential game is not a Nash equilibrium then there is a player i which can improve his utility by an exchange of his strategy.

Definition of potential game $\Rightarrow$

The change of the value of the potential function ϕ is equal to this improvement. I.e., $\phi(Z') < \phi(Z)$ where Z' is the solution after the exchange of the strategy of player i.

Since there are only a finite number of distinct solutions of the potential game, the minimum value of the potential function exists. As observed above, the corresponding solution has to be a Nash equilibrium.

$\blacksquare$

The following theorem bounds for potential games the cost of a best Nash equilibrium in comparison to the cost of an optimal solution.

Theorem 1.6

Let Γ be a potential game with cost function C and potential function Φ. Let Z_{opt} be an optimal solution of Γ. If there is a value α such that $\Phi(Z) \leq C(Z) \leq \alpha \cdot \Phi(Z)$ for all valid solutions Z then there is a Nash equilibrium Z_0 with $C(Z_0) \leq \alpha \cdot C(Z_{opt})$.

Proof:

Let Z_0 be a solution which minimize $\Phi(M)$.

Theorem 1.5 $\Rightarrow$ Z_0 is a Nash equilibrium.

Furthermore there holds

$$\Phi(Z_{opt}) \leq C(Z_{opt}) \leq \alpha \cdot \Phi(Z_{opt}).$$

By the choice of Z_0, we have $\Phi(Z_0) \leq \Phi(Z_{opt})$
Hence,

$$C(Z_0) \leq \alpha \cdot \Phi(Z_0) \leq \alpha \Phi(Z_{opt}) \leq \alpha \cdot C(Z_{opt})$$

■

1.3 Routing games

Load balancing games do not consider the way of the tasks to the servers. Routing games consider the way of the tasks to the servers. We shall investigate routing games on graphs which can be considered as a simple modell of routing in the Internet.

Routing game.

Input:
- directed graph $G = (V, E)$,
- k source/sink-pairs $s_i, t_i \in V$
- for every edge $e \in E$ a mapping $d_e : \mathbb{N} \to \mathbb{Q}$.

Output: For each pair s_i, t_i, $1 \le i \le k$ a path P_i from the source s_i to the sink t_i.

Quality of the path system:

- $f(e)$ load of the edge e; i.e., the number of paths which use the edge e.
- $d_e(f(e))$ the delay of the edge e.
- $\sum_{e \in P_i} d_e(f(e))$ the delay of the path P_i

To each source/sink-pair $[s_i, t_i]$ correspond a player A_i. The goal of the player A_i is to obtain a path P_i which has a delay as small as possible.

Remark:
The load balancing game with equal tasks is a special case of the routing game. We can show this by the construction of a corresponding routing game.

Given
- m servers s_i, $1 \le i \le m$, $S = \{s_1, s_2, \ldots, s$
- n tasks t_j, $1 \le j \le n$, $S_j \subseteq S$
- load dependent response times

$$r_i : \mathbb{Q} \to \mathbb{Q} \quad , \quad 1 \le i \le m$$

we construct the bipartite graph

$$G = (T \cup \{t\}, S, E)$$

where

$$T := \{t_1, t_2, \ldots, t_n\} \quad , \quad t \notin T \cup S$$

$$E := \{(t_j, s_i) \mid s_i \in S_j\}$$
$$\cup \{(s_i, t) \mid 1 \le i \le m\}$$

$$d_e(f(e)) := \begin{cases} 0 & \text{if } e = (t_j, s_i) \text{ for some } j \\ r_i(f(e)) & \text{if } e = (s_i, t) \text{ for some} \end{cases}$$

The routing game with respect to the source/sink-pairs (t_j, t), $1 \le j \le n$ is equivalent to the load balancing game.

$\square$

<u>Question</u>:

Given a current solution $P_1, P_2, \dots, P_k$, is this set of paths a Nash equilibrium?

To get an answer to this question, we consider a path P_i and another path Q_i from the source s_i to the sink t_i.

The change from the path P_i to the path Q_i has the following consequences for the player S_i:

- The load of the edges in $P_i \cap Q_i$ does not change.
- The load on the edges in $Q_i \setminus P_i$ increases by one.
- The edges in $P_i \setminus Q_i$ do not delay the path of the player S_i.

For all edges $e \in E$, $f(e)$ denotes the load of the edge e before the path exchange $P_i \leftrightarrow Q_i$

$\Delta(d(S_i))$ denotes the change of the delay with respect to the player S_i. Then

$$\Delta(d(S_i)) = \sum_{e \in Q_i \setminus P_i} d_e(f(e)+1) - \sum_{e \in P_i \setminus Q_i} d_e(f(e))$$

The player S_i improves his utility iff $\Delta(d(S_i)) < 0$.

$\Longrightarrow$

The current solution $(P_1, P_2, \dots, P_k)$ is a Nash equilibrium if for all $i \in \{1, 2, \dots, k\}$ for all other paths Q_i the following holds

$$\sum_{e \in P_i \setminus Q_i} d_e(f(e)) \leq \sum_{e \in Q_i \setminus P_i} d_e(f(e)+1).$$

Goal:
Extension of the potential function for the load balance game such that we obtain a potential function for the routing game.

Theorem 1.7

The routing game is a potential game.

Proof:

Let Z be a solution of the given routing game and let $f(e)$, $e \in E$ denote the load of e with respect to the solution Z.

The value of the potential function Φ for the solution Z is defined by

$$\Phi(Z) := \sum_{e \in E} \sum_{k=1}^{f(e)} d_e(k).$$

Claim:

After the path exchange $P_i \Leftrightarrow Q_i$ there holds $\Delta(\Phi) = \Delta(d(S_i))$, where $\Delta(\Phi)$ denotes the change of the value of the potential function.

Characterization of $\Delta(\Phi)$:

- For each edge $e \in P_i \setminus Q_i$, the last summand $d_e(f(e))$ vanishs.
- For each edge $e \in Q_i \setminus P_i$, the summand $d_e(f(e)+1)$ is added.
- Thats all.

$\Rightarrow$

$$\Delta(\Phi) = \sum_{e \in Q_i \setminus P_i} d_e(f(e)+1) - \sum_{e \in P_i \setminus Q_i} d_e(f(e))$$

$$= \Delta(d(S_i)).$$

$S(Z)$ denotes the social value of a solution Z; i.e.,

$$S(Z) = \sum_{i=1}^{k} \sum_{e \in P_i} d_e(f(e)).$$

Lemma 1.1

$$S(Z) = \sum_{e \in E} f(e) \cdot d_e(f(e)).$$

Proof:

Definition $\Rightarrow$

$$S(Z) = \sum_{i=1}^{k} \sum_{e \in P_i} d_e(f(e)).$$

For each edge $e \in E$ there holds:

In the sum above, $d_e(f(e))$ occurs exactly $f(e)$ times.

$\Rightarrow$

$$S(Z) = \sum_{e \in E} f(e) \cdot d_e(f(e)).$$

■

Only monoton increasing delay functions make sence. For such functions there holds

$$\phi(Z) \leq S(Z).$$

Note that for each edge $e \in E$ for monotone increasing delay functions there holds

$$\sum_{k=1}^{f(e)} d_e(k) \leq f(e)\, d_e(f(e)).$$

<u>Question</u>

Is it possible to compute a Nash equilibrium for a routing game in polynomial time?

<u>Theorem 1.8</u>

If in a routing game all sources and also all sinks are equal; i.e., $s = s_1 = s_2 = \ldots = s$ and $t = t_1 = t_2 = \ldots = t_k$ then there is a polynomial time algorithm for the computation of a Nash equilibrium.

<u>Proof.</u>

<u>Idea</u>:

Reduce the problem to a minimum cost network flow problem with one source and one sink. Since this problem has a polynomial time algorithm, the assertion follows.

A <u>general flow network</u> $G = (V, E, u, c, b)$ is a directed graph with

- $c : E \to \mathbb{R}$, c_{ij} <u>cost of the edge $(i,j) \in E$</u>
- $u : E \to \mathbb{R}^+$, u_{ij} <u>capacity of the edge</u> <u>$(i,j) \in E$</u>

- $b : V \to \mathbb{R}$ where

$$b(i) = \begin{cases} \underline{\text{supply}} \text{ of node } i & \text{if } b(i) \geq 0 \\ \underline{\text{demand}} \text{ of node } i & \text{if } b(i) \leq 0 \end{cases}$$

<u>Minimum cost network flow problem:</u>

$$\text{minimize} \sum_{(i,j) \in E} c_{ij} x_{ij}$$

where

$$(1) \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = b(i) \quad \forall i \in V$$

(<u>flow balancing condition</u>)

$$(2) \quad 0 \leq x_{ij} \leq u_{ij} \quad \forall (i,j) \in E$$

(<u>capacity condition</u>)

<u>Realization of the idea:</u>

Let be the routing game given by

- the graph $G = (V, E)$
- k equal source/sink-pairs $s,t \in V$
- monoton increasing delay functions $d_e : \mathbb{N} \to \mathbb{Q} \quad \forall e \in E$.

Theorem 1.5 $\Longrightarrow$

There is a valid solution Z_0 such that

$$\phi(Z_0) = \min\{\phi(Z) \mid Z \text{ is valid solution}\}$$

Note that Z_0 is a Nash equilibrium.

We construct the flow network $H = (V, E', u, c, b)$ where

- E' is obtained from E by the replacement of every edge $e \in E$ by k copies $e_1, e_2, \dots, e_k$

$$\begin{aligned} c_{e_i} &:= d_e(i) \\ u_{e_i} &:= 1 \end{aligned} \qquad \text{for } 1 \le i \le k$$

- $$b(w) := \begin{cases} k & \text{if } w = s \\ -k & \text{if } w = t \\ 0 & \text{otherwise.} \end{cases}$$

Since the delay function d_e is monotone increasing there holds:

If a flow of size $h \in \{1, 2, \dots, k\}$ should be send through the copies of an edge e then a minimum cost flow chooses the copies $e_1, e_2, \dots, e_h$.

$\Longrightarrow$

k pairwise edge disjoint path of minimum cost from s to t in H

$\widehat{=}$

Minimum cost flow of size k from s to t in H

$\widehat{=}$

A solution Z_0 of the routing game in G with $\phi(Z_0)$ minimal.

The minimum cost network flow problem can be solved in polynomial time

(see for example:

R.K. Ahuja, T.L. Magnanti, J.B. Orlin: Network Flows: Theory, Algorithms and Applications, Prentice-Hall, 1993.)

■

Question:

Can the method above be extended to the case of k distinct source/sink-pairs $s_i, t_i \in V$, $1 \leq i \leq k$?

Answer:

Yes, but finding a set of k pairwise edge disjoint paths from s_i to t_i, $1 \leq i \leq k$ of minimum cost is NP-complete!

Open problem:
Is the computation of a Nash equilibrium for the routing problem NP-complete?

Further reference:

A. Fabrikant, Ch. Papadimitriou, K. Talwar
The complexity of pure Nash equilibria
Proc. STOC 2004, 604-612.
(can be obtained from the homepage of Alex Fabrikant).

1.4 Some foundations of convex optimization

We shall generalize the routing problem by allowing each player to separate its traffic for sending him on several paths from the source to the sink. For the solution of this generalized routing problem we need some elementary knowledge of convex optimization.

Consider $x, y \in \mathbb{R}^n$. Let $\lambda \in [0,1]$. Each point $z \in \mathbb{R}^n$ with $z = \lambda x + (1-\lambda) y$ is a convex combination of the points x and y. For points $x_1, x_2, \dots, x_t \in \mathbb{R}^n$ each point

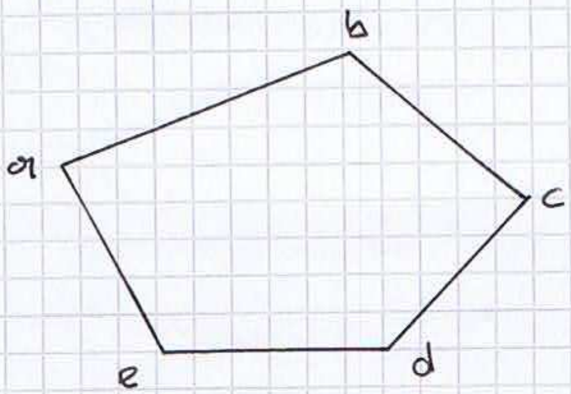
$$z = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_t x_t$$

where $\lambda_i \in [0,1]$, $1 \le i \le t$ and $\sum_{i=1}^{t} \lambda_i = 1$

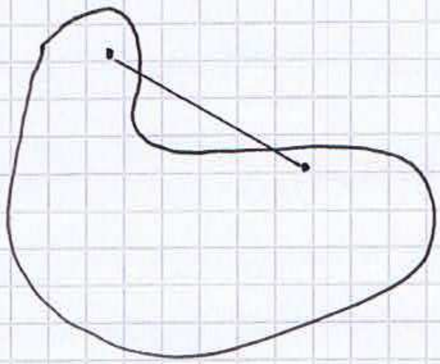
is a <u>convex combination</u> of the points $x_1, x_2, \dots, x_t$.

$K \subseteq \mathbb{R}^n$ is <u>convex</u> if $\forall\, x, y \in K$ also each convex combination of x and y is in K.

<u>Example:</u>



Set of all convex combinations of a, b, c, d, e.

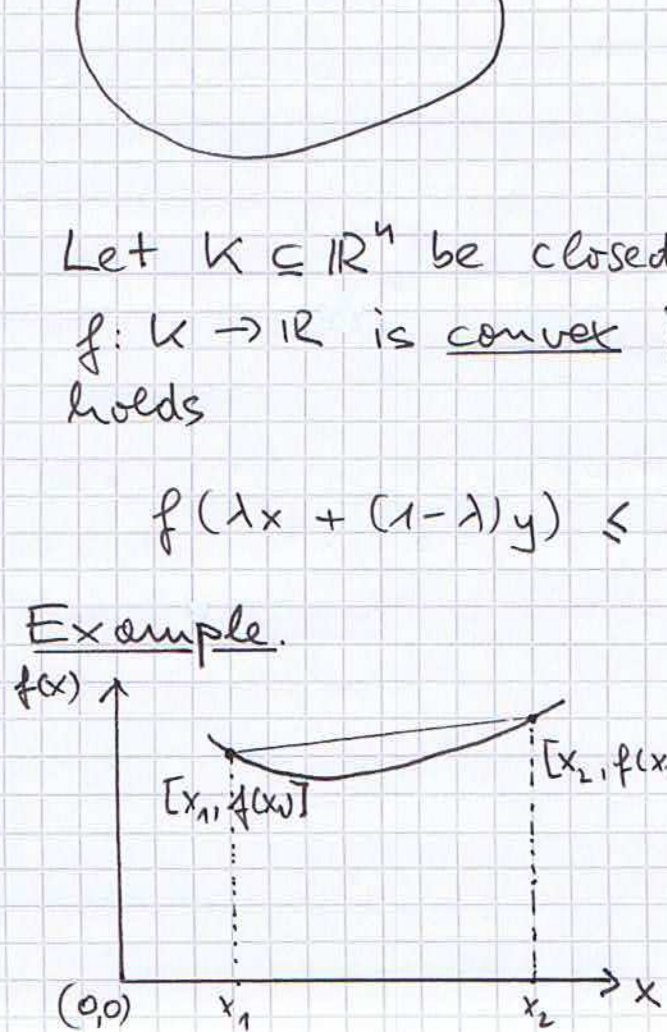


Set which is not convex

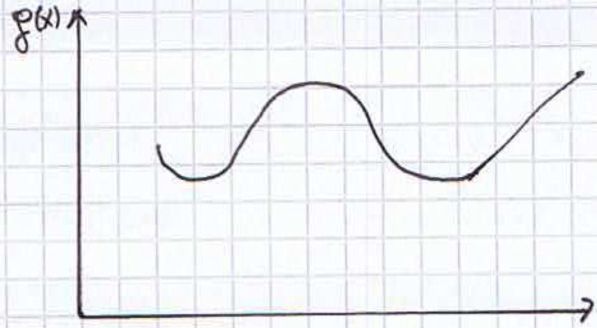
Let $K \subseteq \mathbb{R}^n$ be closed and convex. A function $f: K \to \mathbb{R}$ is <u>convex</u> if for all $x, y \in K$ there holds

$$f(\lambda x + (1-\lambda) y) \leq \lambda \cdot f(x) + (1-\lambda) f(y), \quad 0 \leq \lambda \leq 1$$

<u>Example.</u>



a convex function f.



a nonconvex function g.

Geometric interpretation:

A function $f: K \to \mathbb{R}$ is convex if all points on the line segment joining two arbitrary points $[x_1, f(x_1)]$, $[x_2, f(x_2)]$ of the graph of f are on or above the graph of f.

Theorem 1.9

Let $K \subseteq \mathbb{R}^n$ be closed and convex and let $f_1, f_2, \dots, f_n : K \to \mathbb{R}$ be convex functions. Furthermore let $\alpha_1 \geq 0, \alpha_2 \geq 0, \dots, \alpha_n \geq 0$. Then $f: K \to \mathbb{R}$ with $f(x) := \sum_{i=1}^{n} \alpha_i f_i(x)$ is a convex function.

Proof:

Let $\lambda \in [0,1]$. The following holds:

$$f(\lambda x + (1-\lambda) y) = \sum_{i=1}^{n} \alpha_i f_i(\lambda x + (1-\lambda) y)$$

$$\leq \sum_{i=1}^{n} \alpha_i (\lambda f_i(x) + (1-\lambda) f_i(y))$$

$$= \lambda \cdot \sum_{i=1}^{n} \alpha_i f_i(x) + (1-\lambda) \sum_{i=1}^{n} \alpha_i f_i(y)$$

$$= \lambda f(x) + (1-\lambda) f(y).$$

$\blacksquare$

Exercise:

Prove the following assertions:

a) Let $K \subseteq \mathbb{R}^n$ be a convex set, $f: K \to \mathbb{R}$ be a convex function, $I \subseteq \mathbb{R}$ be a convex set with $f(K) \subseteq I$ and let $g: I \to \mathbb{R}$ be a monotone increasing convex function. Then $g \circ f: K \to \mathbb{R}$ where $g \circ f(x) = g(f(x))$ is a convex function.

b) Let $K \subseteq \mathbb{R}^n$ be convex, $f: K \to \mathbb{R}$ be a convex function and for $\alpha \in \mathbb{R}$ the set $\mathrm{Lev}_\alpha f$ be defined by

$$\mathrm{Lev}_\alpha f := \{ x \in K \mid f(x) \leq \alpha \}.$$

Then $\mathrm{Lev}_\alpha f$ is a convex set $\forall \alpha \in \mathbb{R}$.

c) For $f: K \to \mathbb{R}$ the epigraph $\mathrm{epi}(f)$ of f is defined by

$$\mathrm{epi}(f) := \{ (x,y) \mid x \in K, y \in \mathbb{R} \text{ and } f(x) \leq y \}$$

If $K \subseteq \mathbb{R}^n$ is convex then the following holds: $f: K \to \mathbb{R}$ is convex iff $\mathrm{epi}(f)$ is a convex set.

Let $f: \mathbb{R} \to \mathbb{R}$ be a function. The <u>right-side derivative</u> $f'_r(x_0)$ of f in x_0 is defined by

$$f'_r(x_0) := \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h}.$$

The <u>left-side derivative</u> $f'_\ell(x_0)$ of f in x_0 is defined by

$$f'_\ell(x_0) := \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h}.$$

The <u>derivative</u> $f'(x_0)$ of f in x_0 exists iff $f'_\ell(x_0) = f'_r(x_0)$. Then $f'(x_0) = f'_\ell(x_0)$.

<u>Theorem 1.10</u>

Let $I \subseteq \mathbb{R}$ be an interval, $f: I \to \mathbb{R}$ be a convex function and x_0 be an interior point of I. Then the following hold:

a) $f'_r(x_0)$ and $f'_\ell(x_0)$ exist.

b) f is continuous at the point x_0.

<u>Proof:</u>

exercise

$\blacksquare$

<u>Theorem 1.11</u>

Let $f: I \to \mathbb{R}$ be twice continously differentiable on I. Then f is convex iff $f''(x) \geq 0$ for all $x \in I$.

Proof:

exercise

Let
$$\mathbb{V}^n := \{ \vec{v} = (v_1, v_2, \dots, v_n) \mid v_1, v_2, \dots, v_n \in \mathbb{R} \}$$
be the n-dimensional vector space over $\mathbb{R}$ and let for $\vec{v} \in \mathbb{V}^n$
$$\| \vec{v} \| := \sqrt{\sum_{i=1}^{n} v_i^2}$$
the Euclidian norm.

A vector $\vec{a} \in \mathbb{V}^n$ with $\| \vec{a} \| = 1$ is called a direction in $\mathbb{R}^n$.

The directional derivative $\frac{\partial f}{\partial \vec{a}}(x_0)$ of f in x_0 in direction $\vec{a}$ is defined by
$$\frac{\partial f}{\partial \vec{a}}(x_0) := \lim_{h \to 0^+} \frac{f(x_0 + h\vec{a}) - f(x_0)}{h}.$$

Let $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ be the natural basis vectors of $\mathbb{V}^n$. A function $f: \mathbb{R}^n \to \mathbb{R}$ is called partial differentiable in $x \in D(f)$ with respect to x_k if the directional derivative $\frac{\partial f}{\partial \vec{e}_k}(x)$ exists. In that case, we often write
$$\frac{\partial f}{\partial x_k}(x) \text{ instead of } \frac{\partial f}{\partial \vec{e}_k}(x).$$

f is called differentiable on $E \subseteq D(f)$ if at each point $x \in E$ f is partial differentiable with respect all variables $x_1, x_2, \dots, x_n$. If f is differentiable on E and $\frac{\partial f}{\partial x_k}$ is continuous for all points $x \in E$ for all variables x_k then f is called <u>continuously differentiable</u>.

<u>Theorem 1.12</u>

Let $K \subseteq \mathbb{R}^n$ be convex and let $f : K \to \mathbb{R}$ be a convex function. Then $f(K)$ has at most one local minimum. If there is such a minimum z_0 then this minimum is also global and the set $\{x \mid x \in K, f(x) = z_0\}$ is convex.

<u>Proof</u>:

Assume that there is a local minimum in point $x^0 \in K$.

Definition of local minimum and convexity of f implies

$$f(x^0) \leq f[(1-\lambda)x^0 + \lambda x] \leq (1-\lambda) f(x^0) + \lambda f(x)$$

for all $x \in K$ and sufficient small $\lambda > 0$.

$\Rightarrow$

$$\lambda f(x^0) \leq \lambda f(x).$$

Since $\lambda > 0$ we obtain: $f(x^0) \leq f(x)$

$\Rightarrow$

$z_0 := f(x^0)$ is a global minimum.

Let $x^1 \in K$ be another point with $f(x^1) = z_0$.
Then for $0 \le \lambda \le 1$ there holds

$$z_0 \le f[(1-\lambda)x^0 + \lambda x^1] \le (1-\lambda) f(x^0) + \lambda f(x^1) = z_0$$

$\Rightarrow$

$$f[(1-\lambda)x^0 + \lambda x^1] = z_0$$

$\Rightarrow$

$$\{ x \mid x \in K \text{ and } f(x) = z_0 \} \text{ is convex.}$$

Theorem 1.13

Let K be that set of points $\vec{x}$ of an Euclidian vector space $\mathbb{E}_n$ which fulfill

$$g_i(\vec{x}) \le 0 \qquad i = 1, 2, \ldots, m$$
$$\vec{x} \ge \vec{0}$$

where the functions g_i, $1 \le i \le m$ are convex.
If $K \neq \emptyset$ then K is a convex set.

Proof:

For arbitrary points $\vec{x}_1, \vec{x}_2 \in K$ and $0 \le \lambda \le 1$ let

$$\vec{x} := \lambda \vec{x}_1 + (1-\lambda)\vec{x}_2.$$

We have to show that $\vec{x} \in K$; i.e.,

$$g_i(\vec{x}) \le 0 \quad \text{for } i = 1, 2, \ldots, m$$
and
$$\vec{x} \ge \vec{0}$$

Note that $\lambda \geq 0$, $(1-\lambda) \geq 0$ and $\vec{x}_1, \vec{x}_2 \geq \vec{0}$ imply

$$\vec{x} \geq \vec{0}.$$

Furthermore, for $1 \leq i \leq m$ we obtain

$$\begin{aligned} g_i(\vec{x}) &= g_i[\lambda \vec{x}_1 + (1-\lambda)\vec{x}_2] \\ &\leq \lambda g_i(\vec{x}_1) + (1-\lambda) g_i(\vec{x}_2) \\ &\leq [\lambda + (1-\lambda)] \cdot \max\{g_i(\vec{x}_1), g_i(\vec{x}_2)\} \\ &\leq 0 \end{aligned}$$

■

Consider the following general programming problem:

$$\begin{aligned} &\text{minimize } f(\vec{x}) \\ &\quad g_i(\vec{x}) \leq 0 \qquad 1 \leq i \leq m \\ &\quad \vec{x} \geq \vec{0} \end{aligned}$$

Theorems 1.12 and 1.13 $\Longrightarrow$

If f, g_i, $1 \leq i \leq m$ are convex functions then a valid local minimum is also a global minimum.

Let $K \subseteq \mathbb{E}_n$ and let $f: K \to \mathbb{R}$ be continuously differentiable. For each $x_0 \in K$ the <u>gradient</u> $\nabla f(x_0)$ of f at x_0 is defined by

$$\nabla f(x_0) := \begin{pmatrix} \frac{\partial f}{\partial x_1}(x_0) \\ \frac{\partial f}{\partial x_1}(x_0) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x_0) \end{pmatrix}$$

We shall investigate the relations of convexity, gradients and minima of a convex function.

Theorem 1.14

Let $K \subseteq \mathbb{E}_n$ be an open, convex set and let $f: K \to \mathbb{R}$ be differentiable. Then f is convex iff $\forall x_1, x_2 \in K$ there holds

$$f(x_1) - f(x_2) \geqslant (x_1 - x_2)^T \nabla f(x_2).$$

Proof:

„$\Leftarrow$"

Assume that

$$f(x_1) - f(x_2) \geqslant (x_1 - x_2)^T \nabla f(x_2) \quad \forall x_1, x_2 \in K$$

We have to show that f is convex.

Let $x_1, x_2 \in K$ be arbitrary points and $0 \leq \lambda \leq 1$.
Let

$$x_3 := \lambda x_1 + (1-\lambda) x_2.$$

K convex $\Rightarrow$ $x_3 \in K$.

Assumption $\Rightarrow$

$$(*) \quad f(x_1) - f(x_3) \geq (x_1 - x_3)^T \nabla f(x_3)$$

and

$$(**) \quad f(x_2) - f(x_3) \geq (x_2 - x_3)^T \nabla f(x_3)$$

After multiplication of $(*)$ with λ, multipli= cation of $(**)$ with $(1-\lambda)$ and the addition of the two resulting inequalities, we obtain

$$\lambda f(x_1) - \lambda f(x_3) + (1-\lambda) f(x_2) - (1-\lambda) f(x_3)$$
$$\geq \lambda (x_1 - x_3)^T \nabla f(x_3) + (1-\lambda)(x_2 - x_3)^T \nabla f(x_3)$$

$\Longleftrightarrow$

$$\lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\geq f(x_3) + [\lambda x_1^T + (1-\lambda) x_2^T] \nabla f(x_3) - x_3^T \nabla f(x_3)$$

After the substitution of x_3 by $\lambda x_1 + (1-\lambda) x_2$ we obtain

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2)$$

Hence, f is convex.

"$\Rightarrow$"

Assume that f is convex.

We have to show that

$$f(x_1) - f(x_2) \geq (x_1 - x_2)^T \nabla f(x_2) \quad \forall x_1, x_2 \in K.$$

For arbitrary $x_1, x_2 \in K$, $0 \leq \lambda \leq 1$ the convexity of f implies

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2)$$

$\Longleftrightarrow$

$$\lambda f(x_1) - \lambda f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2) - f(x_2)$$

$\Longleftrightarrow$

$$f(x_1) - f(x_2) \geq \frac{f(x_2 + \lambda(x_1 - x_2)) - f(x_2)}{\lambda}$$

By the definition of directional derivative $\frac{\partial f}{\partial (x_1 - x_2)}(x_2)$ of f in x_2 in direction $(x_1 - x_2)$ we obtain by taking the limit $\lambda \to 0$ at the right side of the inequality above

$$f(x_1) - f(x_2) \geq \frac{\partial f}{\partial (x_1 - x_2)}(x_2)$$

Since $\frac{\partial f}{\partial \vec{a}}(x_0) = \vec{a}^T \nabla f(x_0)$ it follows

$$f(x_1) - f(x_2) \geq (x_1 - x_2)^T \nabla f(x_2).$$

∎

Theorem 1.15

Let $K \subseteq \mathbb{E}_n$ be convex, $f: K \to \mathbb{R}$ be convex and continuously differentiable and $x_0 \in K$.

Then the following holds:

$$f(x_0) \leq f(x) \ \forall x \in K \text{ iff } (x-x_0)^T \nabla f(x_0) \geq 0 \ \forall x \in$$

Proof:

„$\Rightarrow$"

Assume that $f(x_0) \leq f(x) \ \forall x \in K$.

If x_0 is an interior point of K we know from calculus that

$$\nabla f(x_0) = \vec{0}.$$

Hence, $(x-x_0)^T \nabla f(x_0) = 0$.

The assumption above implies for an arbitrary minimal point x_0

$$f(x_0) \leq f(\lambda x + (1-\lambda) x_0) \quad \forall x \in K, \ 0 \leq \lambda \leq 1$$

Hence, we obtain for $\lambda > 0$

$$\frac{f(x_0 + \lambda(x-x_0)) - f(x_0)}{\lambda} \geq 0.$$

If we take at the left side the limit $\lambda \to 0$ then we obtain

$$(x-x_0)^T \nabla f(x_0) \geq 0.$$

„$\Leftarrow$"

Assume that

$$(x-x_0)^T \nabla f(x_0) \geq 0 \quad \forall x \in K.$$

Since f is convex we obtain by Theorem 1.14

$$f(x) - f(x_0) \geq (x-x_0)^T \nabla f(x_0) \geq 0$$

$\Rightarrow$

$$f(x) \geq f(x_0).$$

$\blacksquare$

We are interested in convex functions which are separable. A continuous function $f: \mathbb{R}^n \to \mathbb{R}$ is called <u>separable</u> if f can be written as the sum of n continuous function where each of them depends on a single variable; i.e.,

$$f(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} f_j(x_j),$$

where $f_j: \mathbb{R} \to \mathbb{R}$ are continuous functions.

Note that by Theorem 1.8 the convexity of the function $f_1, f_2, \dots, f_n$ would imply the convexity of the function f.

<u>Theorem 1.16</u>

Let $K \subseteq \mathbb{R}^n$ be a convex set, $c_i: \mathbb{R} \to \mathbb{R}$ be convex, continuously differentiable functions and $\phi: K \to \mathbb{R}$ be defined by

$$\phi(x_1, x_2, \dots, x_n) := \sum_{i=1}^{n} c_i(x_i).$$

Then the following hold:

a) $x^0 \in K$ minimize $\phi(x)$ if for all direction y with $x^0 + \varepsilon y \in K$ for ε small enough holds:

$$\sum_{i=1}^{n} c_i'(y_i) \cdot y_i \geq 0.$$

b) The minimum x^0 exists if K is bounded and closed or $\phi(K)$ is bounded from below.

c) If x^0 and x^1 minimize the function ϕ in K then there exist coefficients a_i, b_i, $1 \leq i \leq n$ such that $\forall x \in [x_i^0, x_i^1]$ there holds $c_i(x) = a_i x + b_i$.

<u>Proof</u>:

exercise

■

Let K be a closed, convex set. A <u>separating algorithm for K</u> decides for a given point x if $x \in K$ or $x \notin K$. If $x \notin K$ then the separating algorithm computes a direction a such that $ax > \max_{y \in K} ay$.

<u>Theorem 1.17</u> (without proof)

Let K be a closed convex set, $c_i : \mathbb{R} \to \mathbb{R}$

be convex, continuously differentiable functions and $\phi : K \to \mathbb{R}$ be defined by $\phi(x_1, x_2, \dots, x_n) := \sum_{i=1}^{n} c_i(x_i)$. If there is a polynomial time separating algorithm for K then a minimum of the function $\phi(x)$ can be approximated in polynomial time

1.5 Nonatomic games

The atomic routing game assumes that each player has to send one unit traffic on one single path from the source to the sink. We obtain another situation if it is allowed to separate the traffic such that each player can send his traffic on several paths from the source to the sink. If we generalize the routing game in that sence then we obtain the following modified routing game:

Given:
- directed graph $G = (V, E)$
- k source / sink - pairs $s_i, t_i \in V$, $1 \le i \le k$
- monotone increasing, continuous functions $d_e : \mathbb{R} \to \mathbb{R}$ $\forall e \in E$.

The following problem has to be solved:

Let P_i, $1 \le i \le k$ be the set of paths from the source s_i to the sink t_i. We have to compute a flow f such that

- $f_P \geqslant 0 \quad \forall P \in \bigcup_{i=1}^{k} P_i$,

  where f_P, $P \in P_i$ denotes that flow which is send from s_i to t_i on P.

  and

- $$\sum_{P \in P_i} f_P = 1.$$

Let

$$f(e) := \sum_{P:\, e \in P} f_P$$

be the total flow on the edge e. The delay $d_P(f)$ of the path P with respect to the flow f is given by

$$d_P(f) := \sum_{e \in P} d_e(f(e)).$$

Goal: The definition of a Nash equilibrium

A feasible flow is a Nash equilibrium if no player can improve the own situation by diversion his flow; i.e.,

$\forall\, i \in \{1, 2, \dots, k\}\ \forall\, P, Q \in \mathcal{P}_i\ \forall\, \delta \in (0, f_P)$:

$$d_P(f) \le d_Q(\tilde{f}),$$

where

$$\tilde{f}_R := \begin{cases} f_R - \delta & \text{if } R = P \\ f_R + \delta & \text{if } R = Q \\ f_R & \text{if } R \notin \{P, Q\} \end{cases}$$

$\tilde{f}$ is obtained by the diversion of δ flow from P to Q.

Since the delay functions are continuous and monotone increasing we obtain the following characterization of a Nash equilibrium:

Lemma 1.2

A flow is a Nash equilibrium iff for all $1 \le i \le k$ for all $P \in \mathcal{P}_i$ with $f_P > 0$ and all $Q \in \mathcal{P}_i \setminus \{P\}$ there holds

$$d_P(f) \le d_Q(f).$$

Proof:

exercise

∎

The characterization given in Lemma 1.2 for a Nash equilibrium is called Wardrop equilibrium. John Glen Wardrop (1886 – 1989) was a pioneer of road traffic research.

Exercise:

Prove that the set of all flow vectors is convex.

Idea:

Use convex optimization for the computation of a Nash equilibrium.

For doing this we need an appropriate objective function. Given a convex function c_e for each edge e, we define the objective function C which is to minimize by

$$C(f) := \sum_{e \in E} c_e(f(e)).$$

C is a convex function on the convex space of all flows. The following Theorem characterizes a minimum flow f_{min} which minimizes $C(f)$ for the case that each function c_e, $e \in E$ is continuously differentiable.

Theorem 1.18

Let c_e, $e \in E$ be continuously differentiable convex functions and let C be defined by $C(f) = \sum_{e \in E} c_e(f(e))$. A flow f_{min} minimizes the function $C(f)$ iff for all $1 \le i \le k$ for all $P \in \mathcal{P}_i$ with $f_{min,P} > 0$ and all $Q \in \mathcal{P}_i$ the following is fulfilled:

$$\sum_{e\in P} c_e'(f_{min}(e)) \leq \sum_{e\in Q} c_e'(f_{min}(e)).$$

<u>Proof</u>:

„$\Rightarrow$"

Assume that f_{min} minimizes $C(f)$ and $\exists i\, \exists P, Q \in \mathcal{P}_i$ such that $f_{min_P} > 0$ and

$$\sum_{e\in P} c_e'(f_{min}(e)) > \sum_{e\in Q} c_e'(f_{min}(e)).$$

Let

$$\delta := f_{min_P}$$

and let the flow $\bar{f}$ be defined by

$$\bar{f}(e) := \begin{cases} f_{min}(e) - \delta & \text{if } e \in P\setminus Q \\ f_{min}(e) + \delta & \text{if } e \in Q\setminus P \\ f_{min}(e) & \text{otherwise} \end{cases}$$

Then

$$(\bar{f} - f_{min}) \cdot \nabla C(f_{min})$$

$$= \delta\Big(\sum_{e\in Q\setminus P} c_e'(f_{min}(e)) - \sum_{e\in P\setminus Q} c_e'(f_{min}(e))\Big)$$

by the definition of partial derivates and the definition of C.

$$\underset{\substack{\uparrow \\ >0}}{=}\ \delta\Big(\underbrace{\sum_{e\in Q} c_e'(f_{min}(e)) - \sum_{e\in P} c_e'(f_{min}(e))}_{<0}\Big)$$

$$< 0$$

a contradiction to Theorem 1.15.

„$\Leftarrow$"

Assume that $\forall i \in \{1, 2, \dots, k\}$ $\forall P, Q \in \mathcal{P}_i$ with $f_{min_P} > 0$ there holds

$$\sum_{e \in P} c_e'(f_{min}(e)) \leq \sum_{e \in Q} c_e'(f_{min}(e))$$

but f_{min} does not minimize the function $C(f)$.

Theorem 1.15 $\Rightarrow$

$\exists$ flow $\bar{f} \neq f_{min}$ with

$$(\bar{f} - f_{min})^T \nabla C(f_{min}) < 0.$$

The following algorithm constructs successively starting with f_{min} the flow $\bar{f}$.

(1) $\tilde{f} := f_{min}$;
(2) <u>while</u> $\tilde{f} \neq \bar{f}$
<u>do</u>

- choose $i \in \{1, 2, \dots, k\}$ with $\exists\, R, Q \in \mathcal{P}_i$ such that $\tilde{f}_R > \bar{f}_R$ and $\tilde{f}_Q < \bar{f}_Q$
- construct from $\tilde{f}$ as follows the flow f':

$$\varepsilon := \min\{\tilde{f}_R - \bar{f}_R,\ \bar{f}_Q - \tilde{f}_Q\}.$$

reduce the flow on R by ε ;
increase the flow on Q by ε ;
$\tilde{f} := f'$

od.

After the termination there holds $\tilde{f} = \bar{f}$.

Let t be the number of iterations of the body of the while-loop. Let R_j and Q_j be the paths chosen during the j-th iteration. ε_j is the flow value transfered from R_j to Q_j.

Goal:
Construction of a contradiction by showing that $(\bar{f} - f_{min})^T \nabla C(f_{min}) \geq 0$.

Let

$f_0 := f_{min}$ and
$f_j := \tilde{f}$ after the j-th iteration, $0 < j \leq t$.

Then $f_t = \bar{f}$ and hence

$$(\bar{f} - f_{min})^T \nabla C(f_{min})$$
$$= \left(\sum_{j=1}^{t} f_j - f_{j-1} \right)^T \nabla C(f_{min})$$
$$= \sum_{j=1}^{t} (f_j - f_{j-1})^T \nabla C(f_{min})$$

Analogously to „$\Rightarrow$", we obtain

$$= \sum_{j=1}^{t} \underset{>0}{\varepsilon_j} \underbrace{\left(\sum_{e \in Q_j} c_e'(f_{min}(e)) - \sum_{e \in R_j} c_e'(f_{min}(e)) \right)}$$

≥ 0.

This contradicts the choice of $\bar{f}$.

■

We shall use Theorem 1.18 for the proof that nonatomic routing games have always a Nash equilibrium exists. For doing this we define also a potential function.

The potential function ϕ for the atomic routing game was the following:

Z feasible solution of the atomic routing game

$f(e)$ load of the edge e (note that $f(e) \in \mathbb{N}_0$)

$$\phi(Z) = \sum_{e \in E} \sum_{k=1}^{f(e)} d_e(k).$$

The straightforward extension of this potential function to nonatomic routing games is the following:

$$\phi(f) := \sum_{e \in E} \int_0^{f(e)} d_e(x)\, dx.$$

<u>Lemma 1.3</u>

Let for all edges e the delay function d_e be continuous and monotone increasing. Let the cost function c_e be defined by

$$c_e(y) := \int_0^y d_e(x)\, dx.$$

Then c_e is continuously differentiable and convex.

Proof:

exercise

■

Theorem 1.19

Let f^* be the flow which minimize the potential function $\phi(f)$. If the delay function d_e is continuous and monotone increasing for all edges e then f^* is a Nash equilibrium for the non-atomic routing game.

Proof:

Lemma 1.3 $\Rightarrow$

c_e with $c_e(y) := \int_0^y d_e(x)\, dx$ is continuously differentiable and convex.

Theorem 1.16 $\Rightarrow$

ϕ has a minimum.

Theorem 1.18 $\Rightarrow$

A flow f^* minimize $\phi(f)$ iff

$\forall\, 1 \leq i \leq k\ \forall P \in \mathcal{P}_i$ with $f^*_P > 0\ \forall Q \in \mathcal{P}_i$

there hold

$$\sum_{e \in P} d_e(f^*(e)) \leq \sum_{e \in Q} d_e(f^*(e)).$$

$\Longrightarrow$
Le 1.2

f^* is a Nash equilibrium.

■

Exercise
Construct a polynomial separation algorithm for the convex set of feasible flows.

Question:
What is the social value for a feasible solution f of the nonatomic routing game?

Answer:

Let the cost function C denote the social value. Then we have

$$C(f) = \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_P \, d_P(f)$$

$$= \sum_{P} f_P \, d_P(f)$$

$$= \sum_{P} f_P \cdot \Big(\sum_{e \in P} d_e(f(e)) \Big)$$

$$= \sum_{e \in E} f(e) \cdot d_e(f(e)).$$

Note that $C(f) = \sum_{e \in E} f(e)\, d_e(f(e))$ is separable.

If we wish to optimize the social value of the solution; i.e., to minimize

$$C(f) = \sum_{e \in E} f(e)\, d_e(f(e))$$

then it makes sence to associate with each edge e the cost function c_e where

$$c_e(x) := x \cdot d_e(x).$$

The following theorem characterizes a feasible flow which is optimal with respect to its social value.

Theorem 1.20

For each edge e let the function d_e be continuously differentiable and the function c_e, defined by $c_e(x) := x\, d_e(x)$, be convex. Furthermore let the function C be defined by

$$C(f) := \sum_{e \in E} c_e(f(e)).$$

Then a flow f_{min} minimizes $C(f)$ iff for all $1 \le i \le k$ for all $P \in \mathcal{P}_i$ with $f_{min\,P} > 0$ and for all $Q \in \mathcal{P}_i$ the following is fulfilled.

Proof: exercise

$$\sum_{e \in P} \left[d_e(f_{min}(e)) + f_{min}(e) \cdot d_e'(f_{min}(e)) \right]$$

$$\leq \sum_{e \in Q} \left[d_e(f_{min}(e)) + f_{min}(e) \cdot d_e'(f_{min}(e)) \right].$$

Proof:

Because of $c_e'(x) = d_e(x) + x\, d_e'(x)$ the assertion follows directly from Theorem 1.18.

Exercise:

Prove ($D(f) \subseteq \mathbb{R}^+$, $f(x) \geq 0 \ \forall x \in D(f)$) and f convex) $\Rightarrow$ $x \cdot f(x)$ is convex.

Lemma 1.2 and Theorem 1.20 imply the following corollary.

Corollary 1.1

For each edge e let the function d_e be continuously differentiable and the function c_e, defined by $c_e(x) := x\, d_e(x)$, be convex. Furthermore let the function C be defined by $C(f) := \sum_{e \in E} c_e(f(e))$. Then a flow f_{min} minimizes $C(f)$ iff

f_{min} is a Nash equilibrium in the same graph with respect to the cost function

$d_e^*: \mathbb{R} \to \mathbb{R}$, $e \in E$ where

$d_e^*(x) := c_e'(x) = d_e(x) + x\, d_e'(x)$.

Proof: exercise