2. The price of anarchy for nonatomic routing games

References

- T. Roughgarden, É. Tardos, How bad is Selfish Routing?, JACM 49, 236 – 259 (2002).
- T. Roughgarden, The Price of Anarchy is independent of the Network Topology, JCSS 67, 341 – 364 (2003).
- R. Feldmann, M. Gairing, T. Lücking, B. Monien, M. Rode, Selfish Routing in Non-cooperative Networks: A Survey, MFCS 2003, LNCS 2747, 21 – 45 (2003).
- T. Roughgarden, Selfish Routing and the Price of Anarchy, MIT Press (2005).

Networks with noncooperating users without central control can be modelled using non-cooperative games. The Internet is a popular example for such a network. The goal of every user is to minimize his own cost without respect to the cost of the other users. Often this contradicts the social value of the solution. The social value might be a measure for the global value of a solution.

Question:

What is the price for the missing of a central control?

This means: What is the price of anarchy?

The usual measure for the price of anarchy is the so-called <u>coordination ratio</u>. This is the ratio of the social values of a worst case Nash equilibrium and an optimal solution.

For nonatomic routing games, the social value $C(f)$ of a feasible solution f is defined by

$$C(f) := \sum_{e \in E} f(e) \cdot d_e(f(e)).$$

We shall investigate the coordination ratio of nonatomic routing games. Before doing this, let us consider our definition of a Nash equilibrium for nonatomic routing games again.

- "A feasible flow is a Nash equilibrium if no player can improve the own situation by diversion his flow; i.e.,

$$\forall i \in \{1,2,\dots,k\} \ \forall P, Q \in \mathcal{P}_i \ \forall \delta \in (0, f_P]$$

$$d_P(f) \leq d_Q(\tilde{f})$$

where

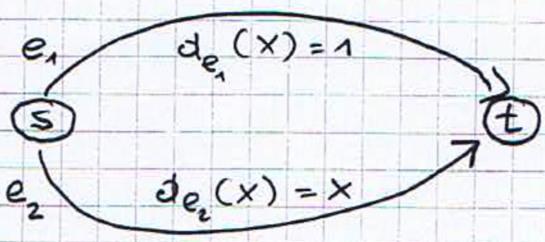
$$\tilde{f}_R := \begin{cases} f_R - \delta & \text{if } R = P \\ f_R + \delta & \text{if } R = Q \\ f_R & \text{if } R \notin \{P, Q\}. \end{cases}$$

Interpretation:

Each portion of the flow of player i wish to optimize his own solution.

If the situation is that player i has to send his flow optimizing a certain cost function, what a Nash equilibrium is depends on this cost function.

Example 2.1 (Pigou)



A flow of size one has to be sent from s to t.

With respect to our definition of a Nash equilibrium for nonatomic routing games, sending the whole flow on edge e_2 is a Nash equilibrium.

Assume that Player i has to send flow of size one from s to t minimizing

a) $\sum_{P \in \mathcal{P}_i} f_P \cdot d_P(f)$

b) $\max_{P \in \mathcal{P}_i} f_P \cdot d_P(f)$.

If Player i minimizes the first cost function then a Nash equilibrium f^* would be

$$f^*(e_1) = f^*(e_2) = \frac{1}{2}.$$

If Player i minimizes the second cost function then a Nash equilibrium f^* would be

$$f^*(e_1) = 1 - f^*(e_2)$$

where $f^*(e_2)$ is the solution of

$$x^2 + x = 1$$

$\diamond$

24.04.

Question

What is the maximal possible value of the coordination ratio?

We shall extend Pigou's example to show that the coordination ratio can be arbitrary large. First, we shall compute the coordination ratio of Pigou's example

Example 2.1 (continued)

Nash equilibrium: $f(e_1) = 0$, $f(e_2) = 1$

$\rightsquigarrow$

$$C(f) = 0 \cdot 1 + 1 \cdot 1 = 1.$$

Theorem 1.20 $\Longrightarrow$

A flow f^* is optimal iff

$$d_{e_1}(f^*(e_1)) + f^*(e_1) \cdot d'_{e_1}(f^*(e_1))$$
$$= d_{e_2}(f^*(e_2)) + f^*(e_2) \cdot d'_{e_2}(f^*(e_2)).$$

Note that

$$d_{e_1}(f^*(e_1)) + f^*(e_1)\, d'_{e_1}(f^*(e_1)) = 1 + f^*(e_1)\cdot 0 = 1$$

and

$$d_{e_2}(f^*(e_2)) + f^*(e_2)\, d'_{e_2}(f^*(e_2)) = f^*(e_2) + f^*(e_2)\cdot 1$$
$$= 2\cdot f^*(e_2)$$

Hence, f^* is optimal iff

$$f^*(e_1) = f^*(e_2) = \frac{1}{2}.$$

For this flow, we obtain

$$C(f^*) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2} = 3/4$$

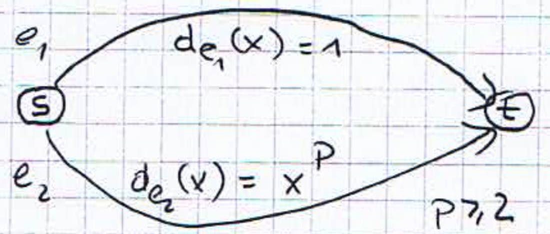
Therefore, we obtain the following coordination ratio:

$$\frac{C(f)}{C(f^*)} = \frac{1}{3/4} = \frac{4}{3}$$

◆

Now we shall extend Pigou's example.

Example 2.2



A flow of size one has to be sent from s to t.

For all $p \geq 2$, we obtain a Nash equilibrium by sending the whole flow f on the edge e_2. For this flow f, we obtain

$$C(f) = 0 \cdot 1 + 1 \cdot 1 = 1$$

Theorem 1.20 $\Rightarrow$

A flow f^* is optimal iff

$$d_{e_1}(f^*(e_1)) + f^*(e_1) d'_{e_1}(f^*(e_1))$$
$$= d_{e_2}(f^*(e_2)) + f^*(e_2) d'_{e_2}(f^*(e_2)).$$

Note that

$$d_{e_1}(f^*(e_1)) + f^*(e_1) d'_{e_1}(f^*(e_1)) = 1$$

and

$$d_{e_2}(f^*(e_2)) + f^*(e_2) d'_{e_2}(f^*(e_2))$$
$$= f^*(e_2)^p + f^*(e_2) \cdot p \cdot f^*(e_2)^{p-1}$$

Hence, f^* is optimal iff

$$f^*(e_2)^p + f^*(e_2) \cdot p \cdot f^*(e_2)^{p-1} = 1$$
$$\Leftrightarrow \quad (p+1) f^*(e_2)^p = 1$$
$$\Leftrightarrow \quad f^*(e_2) = (p+1)^{-\frac{1}{p}}$$

For this flow we obtain

$$C(f^*) = (1 - (p+1)^{-\frac{1}{p}}) \cdot 1 + (p+1)^{-\frac{1}{p}} ((p+1)^{-\frac{1}{p}})^p$$
$$= 1 - (p+1)^{-\frac{1}{p}} + (p+1)^{-1} (p+1)^{-\frac{1}{p}}$$
$$= 1 - \left(1 - \frac{1}{p+1}\right) \cdot (p+1)^{-\frac{1}{p}}$$

Hence,

$$C(f^*) \xrightarrow[p \to \infty]{} 0$$

$\Rightarrow \frac{C(f)}{C(f^*)}$ converges to ∞ if $p \longrightarrow \infty$

<u>Question</u>

In dependence on the possible delay functions, does there exists always a "simple" graph G such that the worst case coordination ratio is obtained by a routing game on G?

The following theorem gives us an answer to this question.

<u>Theorem 2.1</u>

Let D be a class of delay functions such that
 i) D contains the constant functions and
 ii) $x\, d_e(x)$ is convex and continuously differentiable for all functions d_e in D.

Then the worst case of the coordination ratio is obtained by a routing game on a graph $\bar{G} = (\bar{V}, \bar{E})$ where $|\bar{V}| = |\bar{E}| = 2$ and one edge has a constant delay function.

<u>Proof:</u>

<u>Idea:</u>

Start with any worst case routing game R_{wc}. Using R_{wc} construct a routing game on a graph $\bar{G} = (\bar{V}, \bar{E})$ with $|\bar{V}| = |\bar{E}| = 2$ such that its coordination ratio cannot be better than the coordination ratio of R_{wc}.

<u>Realization:</u>

Let $G = (V, E)$ be the graph of the routing game R_{wc} and let $\tilde{f}$ be a worst case Nash equilibrium with respect to the game R_{wc}.

Let $G' = (V, E')$ be the graph which we obtain from G by adding to each edge $e \in E$ a parallel copy e'; i.e.,

$$E' := \{ e, e' \mid e \in E \}.$$

In G', each edge $e \in E$ has the same delay function d_e as in G. The copy e' of $e \in E$ obtains the constant delay function

$$d_{e'}(x) := d_e(\tilde{f}(e)).$$

In G', the same flow as in G has to be sent from the source s_i to the sink t_i, $1 \le i \le k$.

Construction $\Rightarrow$

The optimal flow of the routing game on G' cannot be worse than an optimal flow of the routing game on G.

Let $\bar{f} : E' \to \mathbb{R}$ be defined by

$$\bar{f}(e) := \begin{cases} \tilde{f}(e) & \text{if } e \in E \\ 0 & \text{if } e \in E' \setminus E \end{cases}$$

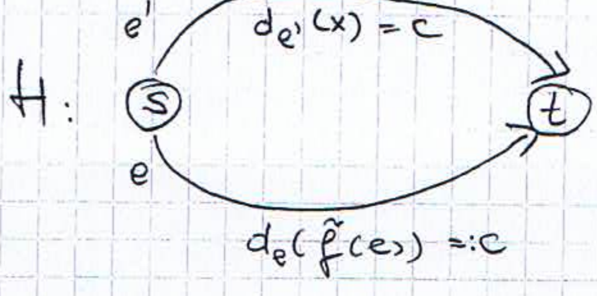
Construction $\Rightarrow$

$\bar{f}$ is a Nash equilibrium of the routing game on G'.

<u>Claim:</u>

The computation of an optimal flow for the routing game on G'.

Let us consider the following situation:



A flow of size $\tilde{f}(e)$ should be sent from s to t.

<u>Goal:</u>

Determination of an optimal flow in H.

The social value $C(\tilde{f})$ of $\tilde{f}$ is the following:

$$C(\tilde{f}) = 0 \cdot c + \tilde{f}(e) \cdot d_e(\tilde{f}(e)) = \tilde{f}(e) \cdot c$$

Theorem 1.20 $\Rightarrow$

A flow f^* is optimal iff

$$d_{e'}(f^*(e')) + f^*(e') \cdot d'_{e'}(f^*(e')) = d_e(f^*(e)) + f^*(e) \cdot d'_e(f^*(e)).$$

Since $d'_{e'}(f^*(e')) = 0$ we obtain

$$d_{e'}(f^*(e')) + f^*(e') \cdot d'_{e'}(f^*(e')) = c.$$

Let $f^*(e)$ be the flow on e in an optimal flow. Then

$$\tilde{f}(e) - f^*(e) = f^*(e')$$

is the flow on e' in an optimal flow f^*.

We can compute $f^*(e)$ by solving the equation

$$d_e(f^*(e)) + f^*(e) \cdot d_e'(f^*(e)) = c.$$

<u>Claim:</u>
From the flow $\bar{f}$ on G' we can compute an optimal flow f^* on G' by dividing for each edge/copy-pair the flow $\bar{f}(e)$ as described above.

<u>Proof of claim:</u>

Consider the following delay functions d_e^*:

$$d_e^*(x) := d_e(x) + x\, d_e'(x).$$

An optimal distribution f^* of the flow $\bar{f}(e)$ to the pair $e, e' \in E'$ implies

$$d_e^*(f^*(e)) = d_{e'}^*(f^*(e')) = c = d_e(\bar{f}(e)).$$

Our goal is to prove that f^* is a Nash equilibrium with respect to the delay functions d_e^*, $e \in E'$.

Assume that f^* is not a Nash equilibrium with respect to the delay functions d_e^*.

Lemma 1.2 $\Longrightarrow$

$\exists\, i \in \{1,2,\dots,k\}$ $\exists\, P \in \mathcal{P}_i$ with $f^*_P > 0$
$\exists\, Q \in \mathcal{P}_i \setminus \{P\}$ such that

$$d^*_P(f^*) > d^*_Q(f^*).$$

This means

$$(*) \quad \sum_{e \in P} d^*_e(f^*(e)) > \sum_{e \in Q} d^*_e(f^*(e))$$

$\Longleftrightarrow$

$$\sum_{e \in P} d_e(f^*(e)) + f^*(e)\, d'_e(f^*(e))$$

$$> \sum_{e \in Q} d_e(f^*(e)) + f^*(e)\, d'_e(f^*(e))$$

For $R \in \mathcal{P}_i$ let $\hat{R}$ be that path from s_i to t_i which we obtain from R if we choose from the edge/copy pair $\{e, e'\}$ with $\{e,e'\} \cap R \neq \emptyset$ always the edge e'.

The construction of f^* and the definition of $d^*_{e'}$

$\Longrightarrow$

$$d^*_{\hat{R}}(f^*) = d^*_R(f^*) \quad \forall\, R \in \mathcal{P}_i$$

Hence, by $(*)$

$$\sum_{e' \in \hat{P}} d^*_{e'}(f^*(e')) > \sum_{e' \in \hat{Q}} d^*_{e'}(f^*(e'))$$

Note that

$$d_{e'}(x) = \underbrace{d_e(\tilde{f}(e))}_{\text{constant.}} \quad \forall e' \in \hat{P} \cup \hat{Q}$$

$\Longrightarrow$

$$\sum_{e'\in\hat{P}} d^*_{e'}(f^*(e')) = \sum_{e'\in\hat{P}} d_{e'}(f^*(e')) + f^*(e') \underbrace{d'_{e'}(f^*(e'))}_{=0}$$

$$= \sum_{e'\in\hat{P}} d_e(\tilde{f}(e))$$

$$= \sum_{e\in P} d_e(\tilde{f}(e))$$

In the same way we obtain

$$\sum_{e'\in\hat{Q}} d^*_{e'}(f^*(e')) = \sum_{e\in Q} d_e(\tilde{f}(e))$$

Therefore, we obtain

$$\sum_{e\in P} d_e(\tilde{f}(e)) > \sum_{e\in Q} d_e(\tilde{f}(e))$$

This is a contradiction to the fact that $\tilde{f}$ is a Nash equilibrium in $G=(V,E)$ with respect to the delay functions d_e, $e\in E$.

$\Rightarrow$

The assumption is false. Hence, f^* is a Nash equilibrium with respect to the delay functions d^*_e, $e\in E'$

$\Longrightarrow$ by Corollary 1.1 $\quad f^*$ is an optimal flow in G' $\square$

Hence, we obtain

$$\frac{C(\bar{f})}{C(f^*)} = \frac{\sum_{e\in E'} \bar{f}(e)\cdot d_e(\bar{f}(e))}{\sum_{e\in E'} f^*(e)\cdot d_e(f^*(e))}$$

$$= \frac{\sum_{e\in E} \bar{f}(e)\cdot d_e(\bar{f}(e))}{\sum_{e\in E'} f^*(e)\cdot d_e(f^*(e))}$$

Exercise:
Prove $\frac{a+b}{a'+b'} \leq \max\{\frac{a}{a'}, \frac{b}{b'}\}$ for $0\leq a, b$, $0 < a', b'$

$$\leq \max_{e\in E} \frac{\bar{f}(e)\cdot d_e(\bar{f}(e))}{f^*(e)\, d_e(f^*(e)) + f^*(e')\, d_{e'}(f^*(e'))}$$

Let $e_0 = (v, w)$ be an edge which maximizes the right side of the inequality above.

Choose $\bar{G} = (\bar{V}, \bar{E})$ where

$\bar{V} = \{v, w\}$, $\bar{E} = \{e, e'\}$ such that

$e = (v, w)$ and e' is the copy of e.

The delay functions d_e and $d_{e'}$ are the same as in G'. A flow of size $\bar{f}(e)$ has to be sent from v to w.

As proved above, the worst case of the coordination ratio with respect to the class D of delay functions is obtained by the routing game defined for $\bar{G}$. This proves the theorem ■

3. Nash's theorem

Up to now we have only considered deterministic strategies and the corresponding Nash equilibria. Such strategies are called pure strategies and such Nash equilibria are called pure. In the first lecture we have seen that the zero-sum-game "head or tail" has no pure Nash equilibrium. The following question arises:

Does randomness help?

As we shall seen, the answer is yes.

Remember, a finite, strategic game $\Gamma = \langle N, (A_i), (u_i) \rangle$ consists of

1. a finite set $N = \{1, 2, \dots, n\}$ of players,
2. for each player $i \in N$ a nonempty, finite set $A_i = \{a_1^i, a_2^i, \dots, a_{m_i}^i\}$ of strategies and
3. for each player $i \in N$ a utility function $u_i : A \longrightarrow \mathbb{R}$ where $A = A_1 \times A_2 \times \dots \times A_n$.

The goal of each player is the maximization of his utility.

A pure strategy of a player i is the deterministic choice of one of his possible strategies in A_i. A mixed strategy x_i for player i is a probability distribution on the set of his pure strategies.

This means that $x_i = (x_i(1), x_i(2), \ldots, x_i(m_i))$ is a vector such that

- $x_i(j) \geq 0$ for $1 \leq j \leq m_i$ and
- $\sum_{j=1}^{m_i} x_i(j) = 1.$

The intuition is that the player i uses his probability distribution x_i for chosing randomly his pure strategy which he play.

Let X_i be the set of all mixed strategies of the player i. Then $X := X_1 \times X_2 \times \ldots \times X_n$ denotes the set of all possible combinations or <u>profiles</u> of mixed strategies.

Let $x = (x_1, x_2, \ldots, x_n) \in X$ be a profile of mixed strategies and $a = (a_1, a_2, \ldots, a_n) \in A$ be a combination of pure strategies. Then

$$x(a) := \prod_{j=1}^{n} x_j(a_j)$$

is the probability of a with respect to the profil x where the players choose their strategies independently.

The <u>expected payoff</u> of the player i with respect to the profil $x = (x_1, x_2, \ldots, x_n) \in X$ is defined by

$$U_i(x) := \sum_{a \in A} x(a) \cdot u_i(a).$$

The goal of each player is the maximization of his expected payoff.

A mixed strategy $x_i \in X_i$ is called <u>pure</u> if $\exists\, j \in \{1, 2, \ldots, m_i\}$ such that

$$x_i(j) = 1 \text{ and } x_i(j') = 0 \quad \forall j' \neq j$$

Such a pure strategy is denoted by $\pi_{i,j}$.

For $x = (x_1, x_2, \ldots, x_n) \in X$ let

$$x_{-i} := (x_1, x_2, \ldots, x_{i-1}, \phi, x_{i+1}, \ldots, x_n).$$

For a mixed strategy $y_i \in X_i$ let

$$(x_{-i}, y_i) := (x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$$

A mixed strategy $z_i \in X_i$ is called <u>optimal for the player i with respect to x_{-i}</u> if $\forall\, y_i \in X_i$ the following is fulfilled:

$$u_i(x_{-i}, z_i) \geq u_i(x_{-i}, y_i).$$

A profile $x = (x_1, x_2, \ldots, x_n) \in X$ is a <u>mixed Nash equilibrium</u> if the strategy x_i is optimal for the player i with respect to x_{-i} for all $i \in N$, i.e., $\forall\, i \in N\ \forall\, y_i \in X_i$ there holds

$$u_i(x_{-i}, x_i) \geq u_i(x_{-i}, y_i).$$

<u>Theorem 3.1</u> (Nash 1950)
Each finite strategic game has a mixed Nash equilibrium.

Goal: Proof of Theorem 3.1

The proof uses a very important deep result in topology, the so-called Brouwer fixed-point theorem. In mathematical terms the theorem says:

If a ball (or its topological equivalent) is mapped continuously into itself, then at least one point has to be mapped into itself.

To formulize this, let

$$S_n := \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; x = \sum_{i=1}^{n+1} \lambda_i e_i : \lambda \geq \vec{0} \text{ and } \sum_{i=1}^{n+1} \lambda_i = 1 \right\}$$

where $e_i := (\underbrace{0, 0, \ldots, 0}_{i-1}, 1, 0, 0, \ldots, 0)$ is the unit vector in the i-th direction. This means that S_n is the interior of an n-dimensional simplex.

Theorem 3.2 (Brouwer fixed-point theorem)

Let $f: S_n \to S_n$ be a continuous function. Then there exists $x \in S_n$ with $f(x) = x$.

There are several ways to prove the Brouwer fixed-point theorem. We shall give a purely combinatorial proof which uses the so-called Sperner's lemma.

Emanuel Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, Abh. Math. Sem. Univ. Hamburg 6 (1928) 265-272.

3.1 Sperner's Lemma

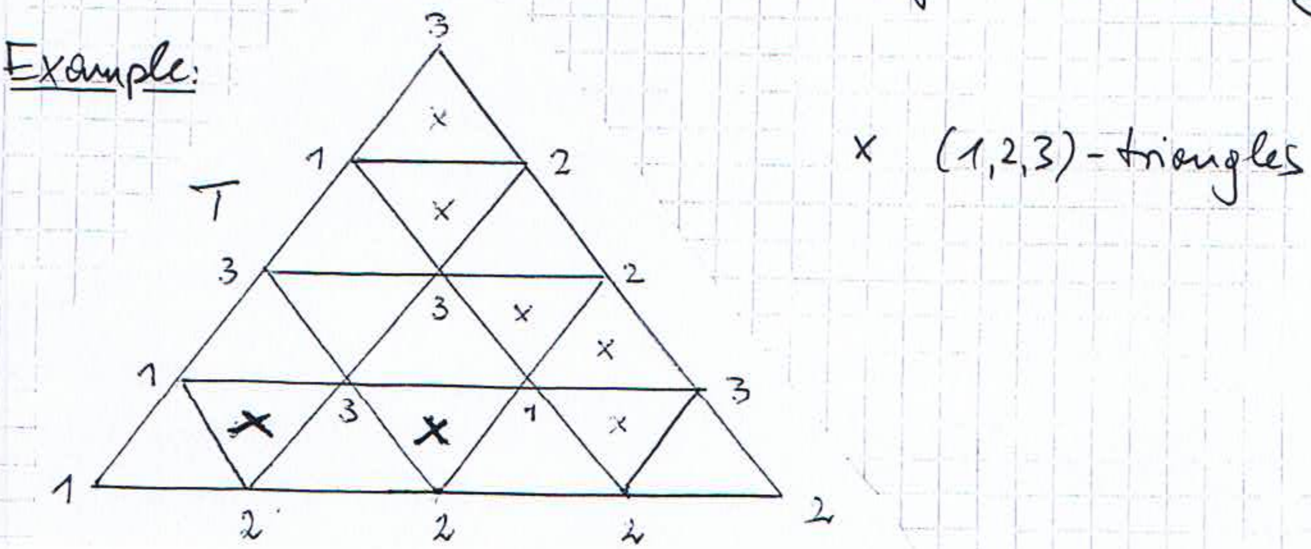
First we shall consider the special case Sperner's lemma for triangles.

Consider a triangle T which is triangulated into many smaller triangles the so-called elementary triangles. The vertices of T are labeled by 1's, 2's and 3's using the following rules:

Sperner-labeling:

1. The three corners of T have different labels.
2. The label of a vertex along any edge of T matches the label of one of the corners spanning that edge.
3. The labels in the interior of T are arbitrary.

Example:



Lemma 3.1 (Sperner's lemma for triangles)

Each Sperner-labelled triangulation of T contains an odd number of elementary (1,2,3)-triangles. This number is at least one.

We shall prove the n-dimensional Sperner's lemma directly. First we need some concepts.

Geometric interpretation of an n-simplex:

0 - simplex $\hat{=}$ a point
1 - simplex $\hat{=}$ a line segment
2 - simplex $\hat{=}$ a triangle
3 - simplex $\hat{=}$ a tetrahedron

$\vdots$

n - simplex $\hat{=}$ an n-dimensional polytope

This means that an n-simplex is the convex hull of $n+1$ affinely independent points in $\mathbb{R}^m$ for $m > n$. These points are the vertices of the simplex. A k-face of an n-simplex is the k-simplex formed by the span of any subset of k vertices.

A triangulation of an n-simplex S is a collection of (distinct) smaller n-simplices whose union is S, with the property that any two of them intersect in a face common to both, or not at all.

The smaller n-simplices are called <u>elementary simplices</u>, and their vertices are called <u>vertices of the triangulation</u>. The $(n-1)$-faces of an n-simplex S are the <u>facets</u> of S. Hence, each n-simplex has $n+1$ facets. As examples, the facets of a line segment are its end points, the facets of a triangle are its sides, and the facets of a tetrahedron are its triangular faces.

Given a triangulation of S, we label the vertices of this triangulation according the following rule:

<u>Sperner-labeling</u>:

1. The $n+1$ vertices of the n-simplex S have distinct labels in $\{1, 2, \ldots, n+1\}$.
2. Each other vertex in the interior of some k-face is labeled with a number of the $k+1$ vertices of S which span this k-face.
3. The labels in the interior of S are arbitrary. All other vertices are labeled arbitrary.

By our construction, a Sperner-labeling on S induces a Sperner-labeling on each facet as $(n-1)$-simplices.

<u>Exercise</u>:
Show that the following rules for a Sperner-labeling are equivalent to the rules above:

The $n+1$ facets of an n-simplex S are numbered by $1, 2, \dots, n+1$. Each node on a facet with number j is labeled with a facet number $\neq j$. In the interior of S, the vertices are labeled by any of the facet numbers.

An elementary n-simplex is called <u>complete labeled</u> if all its $n+1$ vertices have distinct labels.

<u>Lemma 3.2</u> (Sperner's lemma)

Each Sperner-labeled triangulation of an n-simplex S contains an odd number of complete labeled elementary n-simplices. This number is at least one.

There are a lot of possibilities to prove Sperner's lemma. The simplest proofs involve parity arguments and are non-constructive. We shall present a proof based on an inductive argument. One can use the proof for the development of an algorithm for finding a complete labeled elementary n-simplex.

<u>Proof:</u> (by induction on the dimension n)

<u>$n = 1$</u>:

A triangulated 1-simplex is a segmented line. The endpoints of the line have the distinct labels 1 and 2.

Example:

If we run from the endpoint with label 1 to the endpoint with label 2, the kind of the label has to switch an odd number of times. Otherwise, both endpoints would have the same label. Each such a swith induces a complete labeled elementary 1-simplex. This proves the assertion for dimension 1.

Assume that the assertion holds for all dimensions $< n$, $n \geq 1$.

$(n-1) \rightsquigarrow n$:

We consider the n-simplex as a house which is triangulated into many rooms, which are the elementary simplices. A facet of a room is called a door if the n vertices of this facet use all the numbers $1, 2, \dots, n$.

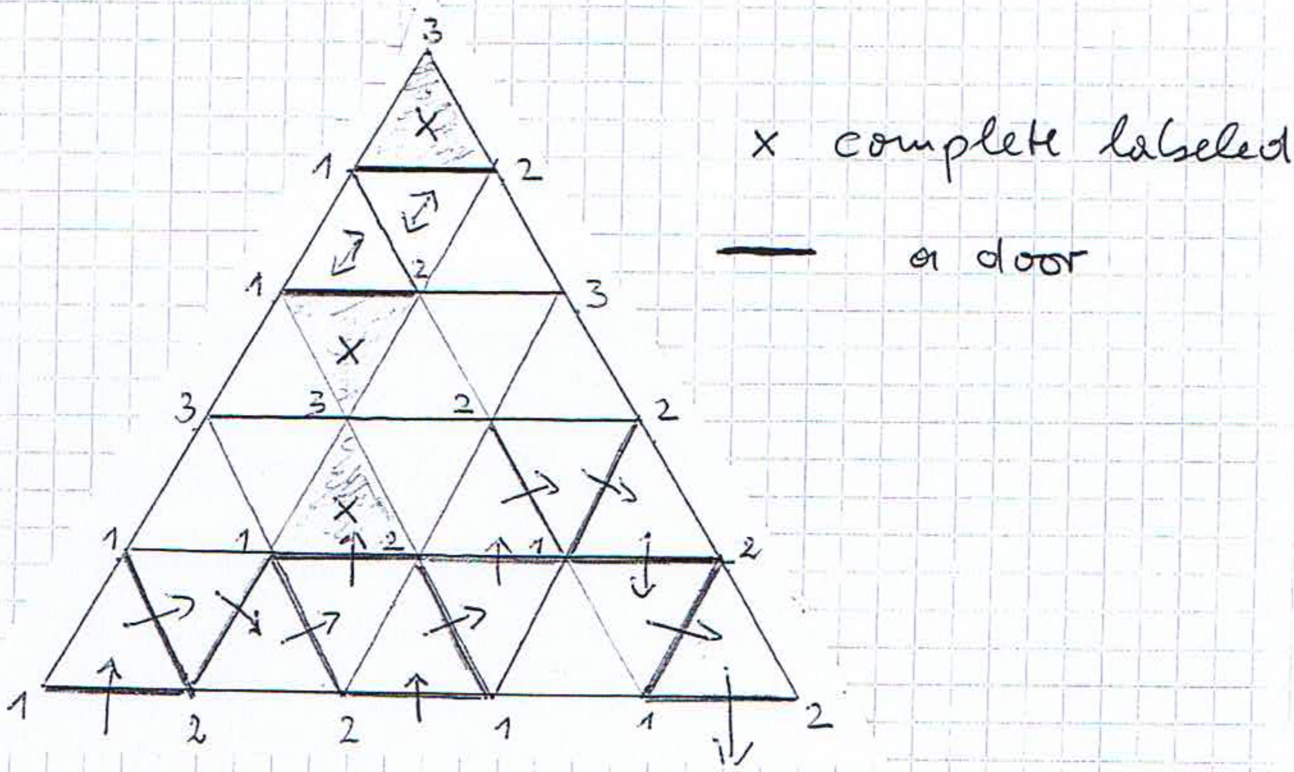
Claim:
The number of doors on the facets of S is at least one and odd.

Proof of claim:

Rules of a Sperner-labeling $\Longrightarrow$

Only the unique facet with the property that its n vertices are marked by $1, 2, \ldots, n$ can contain some doors.

<u>Example</u> (2-simplex)



◆

inductive hypothesis $\Rightarrow$

This facet contains an odd number ≥ 1 of complete labeled elementary $(n-1)$-simplices. These are exactly the doors on this facet.

This proves the claim.

$\square$

Let us consider a room which has at least one door. Two cases can arise:

a) The room is complete labeled.

Then the room has exactly one door.

b) The room is not complete labeled.

Then exactly one of the labels $1, 2, \ldots, n$ appears twice. Hence, this room has exactly two distinct doors.

Altogether, we have shown that the number of doors of a room is at most two. This number is exactly one iff the room is complete labeled.

Now we can prove the assertion using the following <u>trapdoor argument</u>:

Let F be that facet of S which contains some doors.

- If there is any door on F which has not been used before, use such a door and enter the corresponding room.

Either this room is complete labeled or the room has another door, a trapdoor, which we can use. If there is such a trapdoor use this trapdoor and continue until

(+) the entered room has no trapdoor; i.e., the room is complete labeled

or

(++) using a door on F, we have left the n-simplex S.

Repeat the whole until all doors on F are used.

Properties:

1. Since each room has at most two doors, no room is visited twice.
2. Since there are only a finite number of rooms, the walk described above terminates.
3. The paths on the walk form the following pairs:

   Termination of the path because of (+):

   $\begin{cases} 1 \text{ door on } F \\ 1 \text{ complete labeled elementary } n\text{-simplex} \end{cases}$

   Termination of the path because of (++):

   $\begin{cases} 1 \text{ door on } F \\ 1 \text{ door on } F \end{cases}$

4. The complete labeled elementary n-simplices not reached during the walk form pairs. Hence, its number is zero or even.

   Proof as exercise

Altogether, we obtain

Claim and property 3 $\Longrightarrow$

The number of complete labeled elementary n-simplices reached during the walk is at least one and odd

$\Longrightarrow$

4. The number of complete labeled elementary n-simplices in the triangulation of S is at least one and odd.

$\blacksquare$

The trapdoor argument to prove Sperner's lemma is due to Cohen and Kuhn:

D. I. A. Cohen, On the Sperner lemma, J. Combin. Theory 2 (1967), 585 - 587.

H. W. Kuhn, Simplicial Approximation of Fixed Points, Proc. Nat. Acad. Sci. U.S.A. 61 (1968), 1238 - 1242.

The presentation above is taken from

Francis Edward Su, Rental Harmony: Sperner's Lemma in Fair Division, Amer. Math. Monthley 106 (1999), 930 - 942.

Now we can prove the Brouwer fixed-point theorem.

3.2 The Brouwer fixed-point theorem

Theorem 3.2 (Brouwer fixed-point theorem)

Let $f: S_n \to S_n$ be a continuous function. Then there exists $x \in S_n$ such that $f(x) = x$.

Proof:

Consider any $x \in S_n$. If $x = f(x)$ then nothing is to prove.

Assume that $x \neq f(x)$.

Consider x and $f(x)$ as convex combinations of the vertices of S_n, i.e., of the $n+1$ unit vectors. This means

$$x = \sum_{i=1}^{n+1} \lambda_i e_i \quad \text{and} \quad f(x) = \sum_{i=1}^{n+1} \mu_i e_i$$

where $\lambda_i, \mu_i \geq 0$, $1 \leq i \leq n+1$ and

$$\sum_{i=1}^{n+1} \lambda_i = \sum_{i=1}^{n+1} \mu_i = 1.$$

Since $x \neq f(x)$ there is $i \in \{1, 2, \ldots, n+1\}$ with $\lambda_i > \mu_i$.

We assign to x the following unique number $num(x)$

$$num(x) := \min_{1 \leq i \leq n+1} \{ i \mid \lambda_i > \mu_i \}.$$

The idea is to prove the assertion using Sperner's lemma.

Note that $e_1, e_2, \ldots, e_{n+1}$ are the vertices of S_n. If we write e_i, $1 \le i \le n+1$ as convex combination of $e_1, e_2, \ldots, e_{n+1}$ then we obtain

$$e_i = \sum_{j=1}^{n+1} \lambda_j e_j$$

where $\lambda_i = 1$ and $\lambda_j = 0$ for $j \neq i$.

Hence,

$$\text{num}(e_i) = i \quad \text{for } 1 \le i \le n+1.$$

Let us consider the points on a facet F of S_n. Let e_j be the vertex of S_n which is not a vertex of F.

$\Rightarrow$

For all points $x = \sum_{i=1}^{n+1} \lambda_i e_i$ on F there holds

$$\lambda_j = 0.$$

$\Rightarrow$

$$\text{num}(x) \neq j.$$

This observation holds analogously for the facets of F a.s.o.

$\Rightarrow$

Independently of the triangulation of S_n, the numbering above is a Sperner-labeling.

To apply Sperner's lemma we need a triangulation of the n-simplex. Consider the following sequence of triangulations.

- In the k-th triangulation each facet of the simplex is separated into k segments of equal size.

For the vertices of the triangulation we always use the Sperner-labeling defined above.

Sperner's lemma $\Rightarrow$

Each of these triangulations has a complete labeled elementary n-simplex.

Choose in each triangulation such a complete labeled elementary n-simplex. Let x_k be the point in the center of the chosen n-simplex of the k-th triangulation. We obtain a sequence

$$x_1, x_2, \ldots, x_k, \ldots$$

of points.

Since the n-simplex is a bounded and closed set, the sequence above contains a subsequence which has a limit x.

<u>Claim:</u> $f(x) = x$

<u>Proof of claim:</u>

Assume that $f(x) \neq x$.

Let

$$x = \sum_{i=1}^{n+1} \lambda_i e_i \quad \text{and} \quad f(x) = \sum_{i=1}^{n+1} \mu_i e_i$$

where $\lambda_i, \mu_i \geq 0$ for $1 \leq i \leq n+1$ and

$$\sum_{i=1}^{n+1} \lambda_i = \sum_{i=1}^{n+1} \mu_i = 1.$$

$\Longrightarrow$

$$\exists \; j \in \{1, 2, \dots, n+1\} \quad \text{with } \mu_j > \lambda_j$$

Let

$$\varepsilon := \mu_j - \lambda_j .$$

For all points x' in a sufficient small neighbourhood δ_1 of x there holds

$$x' = \sum_{i=1}^{n+1} \lambda_i' e_i \quad \text{with } \lambda_j' < \lambda_j + \varepsilon/2 .$$

f continuous $\Longrightarrow$

For all x' in a sufficient small neighbourhood δ_2 of x there holds

$$f(x') = \sum_{i=1}^{n+1} \mu_i' e_i \quad \text{with } \mu_j' > \mu_j - \varepsilon/2$$

Let

$$\delta := \min\{\delta_1, \delta_2\}.$$

For all points x' in the δ neighbourhood of x the following is fulfilled:

$$\mu_j' - \lambda_j' > (\mu_j - \varepsilon/2) - (\lambda_j + \varepsilon/2)$$
$$= (\mu_j - \lambda_j) - \varepsilon$$
$$= 0$$

$\Rightarrow$

For all points x' in the δ-neighbourhood of x the label of x' is unequal j.

From this we shall derive a contradiction.

The point x is the limit of the centers of simplices of strictly decreasing sizes.

$\Rightarrow$

For k large enough, the whole n-simplex is contained in the δ-neighbourhood of his center x_k.

$\Rightarrow$

The whole n-simplex is in the δ-neighbourhood of x.

This contradicts the fact that the n-simplex is complete labeled; i.e., one vertex of the simplex has the label j.

■

We have proved the Brouwer fixed-point theorem for the case that f is a function from S_n to S_n for an $n \in \mathbb{N}$. This theorem can be generalized to

<u>Theorem 3.3</u> (Brouwer fixed-point theorem)

Let $f: D \to D$ be a continuous function and let D be closed, convex and bounded. Then there exists $x \in D$ such that $f(x) = x$.

<u>Proof:</u>

exercise

$\blacksquare$

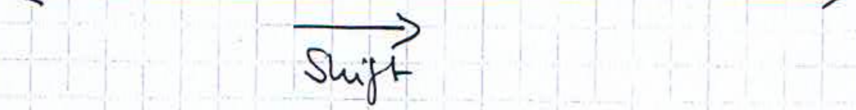
The generalization above is easy to prove if D is the set of convex combinations of a finite set of points, i.e., D is affine isomorphic to an n-simplex S_n, $n \in \mathbb{N}$.

The following examples show that the assumptions bounded, convex and closed are necessary.

<u>Example:</u>

a) <u>not bounded</u>:

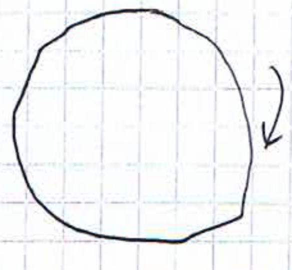
Consider the line, which is a closed and convex but not bounded region in the plane



We can define a function f which performs a shift
$\Rightarrow$ f has no fixed point.

b) <u>not convex</u>

Consider a cycle in the plane. The cycle is closed and bounded but not convex.



rotation

Clearly, a function which defines a rotation has no fixed-point.

c) <u>not closed</u>

Consider an open interval. It is also possible to define a shift.

3.3 The proof of Nash's theorem

<u>Theorem 3.1</u> (Nash 1950)

Each finite strategic game has a mixed Nash equilibrium.

<u>Proof</u>: (Nash 1951)

<u>Idea</u>:

Define a continuous function $f: X \to X$ where $X = X_1 \times X_2 \times \ldots \times X_n$ and prove for $x^* \in X$

$$f(x^*) = x^* \Rightarrow x^* = (x_1^*, x_2^*, \ldots, x_n^*) \text{ is a Nash equi.}$$

Then, the assertion follows from the Brouwer fixed-point theorem.

First we shall characterize exactly the case that a profile $x^* = (x_1^*, x_2^*, \dots, x_n^*) \in X$ is a Nash equilibrium.

Lemma 3.3

A profile $x^* = (x_1^*, x_2^*, \dots, x_n^*) \in X$ is a Nash equilibrium iff for all player i and each pure strategy π_{ij}, $1 \leq j \leq m_i$ there holds

$$u_i(x^*) \geqslant u_i(x^*_{-i}, \pi_{i,j}).$$

Proof:

„$\Rightarrow$"

Assume that x^* is a Nash equilibrium.

Definition $\Rightarrow$

$\forall$ player i $\forall$ $y_i \in X_i$ there holds

$$u_i(x^*_{-i}, x_i^*) \geqslant u_i(x^*_{-i}, y_i).$$

This implies

$\forall$ player i for $1 \leq j \leq m_i$

$$u_i(x^*) \geqslant u_i(x^*_{-i}, \pi_{i,j}).$$

„$\Leftarrow$"

For each mixed strategy $x_i \in X_i$ there holds

$$(*) \quad U_i(x^*_{-i}, x_i) = \sum_{j=1}^{m_i} x_i(a^i_j) \cdot U_i(x^*_{-i}, \Pi_{i,j})$$

We obtain this equality by the following calculation:

$$U_i(x^*_{-i}, x_i) \underset{\text{def}}{=} \sum_{a \in A} (x^*_{-i}, x_i)(a) \cdot u_i(a)$$

$$= \sum_{a \in A} \Big(\Big(\prod_{\substack{e=1 \\ e \neq i}}^{n} x^*_e(a_e) \Big) x_i(a_i) \Big) \cdot u_i(a)$$

$$= \sum_{j=1}^{m_i} \sum_{\substack{a \in A: \\ a_i = a^i_j}} \Big(\Big(\prod_{\substack{e=1 \\ e \neq i}}^{n} x^*_e(a_e) \Big) \cdot x_i(a^i_j) \Big) \cdot u_i(a)$$

$$= \sum_{j=1}^{m_i} x_i(a^i_j) \Big(\sum_{\substack{a \in A: \\ a_i = a^i_j}} \Big(\Big(\prod_{\substack{e=1 \\ e \neq i}}^{n} x^*_e(a_e) \Big) \cdot 1 \Big) u_i(a) \Big)$$

$$= \sum_{j=1}^{m_i} x_i(a^i_j) \Big(\sum_{\substack{a \in A: \\ a_i = a^i_j}} \Big(\Big(\prod_{\substack{e=1 \\ e \neq i}}^{n} x^*_e(a_e) \Big) \cdot \Pi_{i,j}(a^i_j) \Big) u_i(a) \Big)$$

$$= \sum_{j=1}^{m_i} x_i(a^i_j) \Big(\sum_{a \in A} \Big(\Big(\prod_{\substack{e=1 \\ e \neq i}}^{n} x^*_e(a_e) \Big) \Pi_{i,j}(a_i) \Big) u_i(a) \Big)$$

$$= \sum_{j=1}^{m_i} x_i(a^i_j) \cdot U_i(x^*_{-i}, \Pi_{i,j}).$$

Assume

$$U_i(x^*) \geq U_i(x^*_{-i}, \Pi_{i,j})$$

for all players $i \in N$ and all pure strategies $\Pi_{i,j}$, $1 \leq j \leq m_i$.

Since $\sum_{j=1}^{m_i} x_i(a_j^i) = 1$, by assumption and equality (*) it follows

$$u_i(x^*) \geq u_i(x^*_{-i}, x_i)$$

for all players $i \in N$ for all $x_i \in X_i$

$\underset{\text{def.}}{\Longrightarrow}$ x^* is a mixed Nash equilibrium. $\boxtimes$

Goal:

Find a profile $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ such that

$$u_i(x^*_{-i}, \pi_{i,j}) \leq u_i(x^*) \quad \forall i \in N \; \forall \; 1 \leq j \leq m_i$$

or equivalent

$$u_i(x^*_{-i}, \pi_{i,j}) - u_i(x^*) \leq 0 \quad \forall i \in N \; \forall \; 1 \leq j \leq m_i.$$

For $x = (x_1, x_2, \dots, x_n) \in X$ and $1 \leq i \leq n$, $1 \leq j \leq m_i$ let

$$\varphi_{i,j}(x) := \max\{0, u_i(x_{-i}, \pi_{i,j}) - u_i(x)\}.$$

This means that $\varphi_{i,j}(x)$ describes the improvement of player i if he chooses the strategy $\pi_{i,j}$ instead of x_i under the assumption that no other player changes his strategy.

Now we define a function $f: X \to X$ which realizes our idea.

For $x = (x_1, x_2, \dots, x_n) \in X$ let

$$f(x) := (x_1', x_2', \dots, x_n')$$

where for $1 \le i \le n$, $1 \le j \le m_i$

$$x_i'(j) := \frac{x_i(j) + \varphi_{i,j}(x)}{1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x)}.$$

Exercise:

Prove that X is bounded, closed an convex.

The next lemma implies that we can apply the Brouwer fixed-point theorem to f.

Lemma 3.4

a) $x \in X \Rightarrow f(x) = (x_1', x_2', \dots, x_n') \in X$.

b) $f: X \to X$ is continuous.

Proof:

Exercise

$\boxtimes$

Brouwer fixed-point theorem $\Rightarrow$

$\exists\, x^* = (x_1^*, x_2^*, \dots, x_n^*) \in X$ with $f(x^*) = x^*$.

It remains to prove that x^* is a Nash equilibrium. By the definition of f it holds for $1 \le i \le n$, $1 \le j \le m_i$:

$$x_i^*(j) = \frac{x_i^*(j) + \varphi_{i,j}(x^*)}{1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x^*)}$$

$$\Leftrightarrow \quad x_i^*(j) \cdot \left(1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x^*)\right) = x_i^*(j) + \varphi_{i,j}(x^*).$$

$$\Leftrightarrow \quad x_i^*(j) \cdot \sum_{k=1}^{m_i} \varphi_{i,k}(x^*) = \varphi_{i,j}(x^*) \qquad (*)$$

Goal:

Proof that $(*)$ for all j implies $\varphi_{i,j}(x^*) = 0$ for all j.

Lemma 3.5

For each profile $x = (x_1, x_2, \dots, x_n) \in X$ for each player $i \in N$ there exists $j \in \{1, 2, \dots, m_i\}$ such that

$$x_i(j) > 0 \text{ and } \varphi_{i,j}(x) = 0$$

Proof:

We have proved above the following.

$$u_i(x) = \sum_{e=1}^{m_i} x_i(e) \cdot u_i(x_{-i}, \pi_{i,e}).$$

Since $\sum_{e=1}^{m_i} x_i(e) = 1$ it has to exist a $j \in \{1, 2, \dots, m_i\}$ such that

$$x_i(j) > 0 \text{ and } u_i(x_{-i}, \pi_{i,j}) \leq u_i(x).$$

$\Leftrightarrow$

$x_i(j) > 0$ and $u_i(x_{-i}, \pi_{i,j}) - u_i(x) \leq 0$

$\underset{\text{def. of } \varphi_{i,j}(x)}{\Longleftrightarrow} \quad x_i(j) > 0$ and $\varphi_{i,j}(x) = 0$.

$\boxtimes$

Lemma 3.5 $\Rightarrow$

For each player $i \in N$ there exists $j \in \{1, 2, \dots, m_i\}$ such that

$$x_i^*(j) > 0 \text{ and } \varphi_{i,j}(x^*) = 0.$$

The equation $(*)$ implies

$$0 = \varphi_{i,j}(x^*) = x_i^*(j) \cdot \sum_{k=1}^{m_i} \varphi_{i,k}(x^*)$$

Since $\varphi_{i,k}(x^*) \geq 0$ for $1 \leq k \leq m_i$ we obtain

$$\varphi_{i,k}(x^*) = 0 \text{ for } 1 \leq k \leq m_i.$$

$\Rightarrow$

For all player $i \in N$ for all $1 \leq j \leq m_i$ there holds

$$u_i(x^*) \geq u_i(x^*_{-i}, \pi_{i,j}).$$

Lemma 3.3 $\Rightarrow$

x^* is a mixed Nash equilibrium.

This proves Nash's theorem.

$\blacksquare$

Observation:

The proof above implies that for a Nash equilibrium $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ always the following is fulfilled:

$$x_i^*(j) > 0 \Rightarrow U_i(x_{-i}^*, \pi_{i,j}) = U_i(x^*).$$

This implies that the utility of player i remains the same if all other players do not change their strategies and player i switch to strategy $\pi_{i,j}$.

Nash's first proof and also the proofs usually presented in text books don't use Brouwer's fixed-point theorem directly. They use a generalization, Kakutani's fixed-point theorem which can be proved using Brouwer's fixed-point theorem.

Theorem 3.4 (Kakutani fixed-point theorem)

Let $X \subseteq \mathbb{R}^n$ be compact and closed and let $f: X \to 2^X$ be a function which fulfills

i) $\forall x \in X$, $f(x)$ is nonempty and convex and
ii) the graph of f is continuous (i.e., for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ $\forall n$: $(x_n \to x$ and $y_n \to y) \Rightarrow y \in f(x)$.)

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.

We don't prove Kakutani's fixed-point theorem. We use this theorem to give an alternative proof of Nash's theorem.

Proof of Theorem 3.1 (Nash 1950):

For $x = (x_1, x_2, \ldots, x_n) \in X$ and $1 \leq i \leq n$ we define

$$Z_i(x) := \{ z_i \in X_i \mid z_i \text{ is optimal for player } i \text{ with respect to } x_{-i} \}.$$

and

$f : X \to 2^X$ where

$$f(x_1, x_2, \ldots, x_n) = Z_1(x) \times Z_2(x) \times \ldots \times Z_n(x).$$

Exercise:

a) Prove that $f(x)$ is nonempty and convex for all $x \in X$.

b) Prove that the graph of f is continuous.

Kakutani's fixed-point theorem $\Rightarrow$

$\exists\, x^* \in X$ such that $x^* \in f(x^*)$.

Definition of f $\Rightarrow$

x^* is a mixed Nash equilibrium.

■