4. The computation of Nash equilibria

References

Herbert E. Scarf, The Computation of Economic Equilibria, New Haven and London, Yale University Press 1973.
(can be obtained from Scarf's homepage)

Herbert E. Scarf, The Computation of Equilibrium Prices: An Exposition, in K.J. Arrow, M.D. Intriligator (eds.), Handbook of Mathematical Economics, Vol. II, North-Holland, 1982, 1007–1061.
(can be obtained from Scarf's homepage).

Herbert E. Scarf, The computation of equilibria for the Walrasian model: a personal account, in Pascal Bridel (ed.), General Equilibrium Analysis: A Century after Walras, Routledge 2011, 46–73.
(can be obtained from Scarf's homepage)

Richard D. McKelvey, Andrew McLennan, Computation of Equilibria in Finite Games, Handbook of Computational Economics, Elsevier 1996.
(can be obtained from Michael Kearns homepage)

Christos H. Papadimitriou, The Complexity of Finding Nash Equilibria, in Noam Nisan, Tim Roughgarden, Éva Tardos, Vijay V. Vazirani (eds.), Algorithmic Game Theory, Cambridge University Press, 2007, 29 - 51.

<u>Goal</u>:
The development of methods for the computation of Nash equilibria in general n-persons games; i.e., the approximation of a Nash equilibrium.

Proof of the theorem of Nash $\Rightarrow$

A method for the approximation of a Brouwer fixed-point would suffice.

$\leadsto$

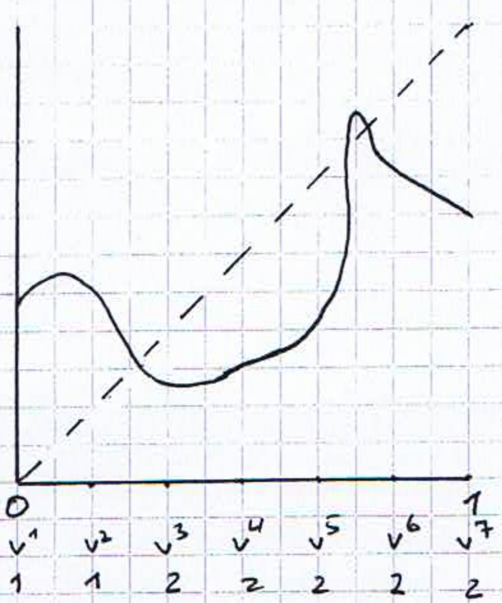
<u>Goal</u>:
The development of a method for the approximation of a Brouwer fixed-point.

The proof of the Brouwer fixed-point theorem which uses Sperner's lemma suggests the following idea:

- Use the Sperner-labeling of the proof of the Brouwer fixed-point theorem, a sufficient fine triangulation of S_n and find a complete labeled elementary n-simplex.

Example (1-simplex):

Consider a continuous function $f: [0,1] \to [0,1]$.



If we interpret the unit interval $[0,1]$ as 1-simplex in $\mathbb{R}^2$ then we obtain the following unique representation of $x \in [0,1]$ as a point $\lambda_1 e_1 + \lambda_2 e_2$ where $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$:

$$\lambda_1 := 1 - x \quad \text{and} \quad \lambda_2 := x.$$

Hence, the point x obtains the following Sperner-label $\mathrm{num}(x)$:

$$\mathrm{num}(x) := \begin{cases} 1 & \text{if } f(x) > x \\ 2 & \text{if } f(x) < x \end{cases}$$

A triangulation of the unit interval is a sequence $0 = v^1 < v^2 < \ldots < v^{k-1} < v^k = 1$ of strongly monotone increasing points.

Construction $\Longrightarrow$

If $num(v^i) \neq num(v^{i+1})$ then there is a fixed-point in the interval $[v^i, v^{i+1}]$.

Note that the other direction is only valid if the triangulation is fine enough. In our example the interval $[v^5, v^6]$ contains two fixed-points but $num(v^5) = num(v^6) = 2$.

If the triangulation is "fine enough" then v^i and v^{i+1} are points which are "not far away". In the case that $num(v^i) \neq num(v^{i+1})$, one of the values $f(v^i) - v^i$ and $f(v^{i+1}) - v^{i+1}$ is positive and the other is negative. A point inside the interval $[v^i, v^{i+1}]$ can be considered as an approximation of a fixed-point.

◆

<u>Question:</u>
How to find a complete labeled elementary n-simplex in S_n?

To get an answer to this question let us consider the proof of Sperner's lemma again. The proof is constructive and suggests the following method for the computation of a complete labeled elementary n-simplex:

- <u>while</u> $\exists$ an unused door on the facet F of S_n

  <u>do</u>

use such a door and enter the corresponding room;

<u>while</u> the entered room has another door

<u>do</u>

use this door

<u>od</u>;

<u>if</u> we are in a complete labeled elementary n-simplex

<u>then</u>

EXIT

<u>fi</u>

<u>od</u>.

We have to answer the following questions:

1) Given a triangulation how to find the doors?

2) How to construct an appropriate triangulation?

For these technical details see the papers of Herbert E. Scarf.

5. <u>Correlated equilibria</u>

The strategy chosen by a player with respect to a mixed Nash equilibrium depends on a probability distribution where the decisions of the players are made independently.

Sometimes it makes sence that the decisions of the players are not independent.

Example:

Bach or Stravinsky (BoS)

	B	S
B	2,1	0,0
S	0,0	1,2

payoff matrix

It would make sence if both player observe one random coin toss ; i.e., with probability 1/2 the outcome is head. If the outcome is head then both players go to Bach. If the outcome is number then both players go to Stravinsky. Performing this procedure results into another equilibrium.

Both players observe the same random variable and have perfectly correlated information.

More generally, their information may be less than perfectly correlated. Consider the following experiment:

· There are three possible outcomes x, y and z.

- Player 1 knows that the outcome is either x or an element of $\{y, z\}$.
- Player 2 knows that the outcome is either an element of $\{x, y\}$ or z.

$\Rightarrow$

We obtain the following partitions of information:

Player 1: $[\{x\}, \{y, z\}]$
Player 2: $[\{x, y\}, \{z\}]$.

$\leadsto$

A strategy of player 1 consists of two actions:

one action if the outcome is x and another action if the outcome is an element of $\{y, z\}$.

Analogously, a strategy of player 2 consists of two actions.

The strategy of a player is <u>optimal</u> if, given the strategy of the other players, for each outcome with respect to his information partition he cannot perform a better action as that action given by his strategy.

To illustrate how a player uses his information in choosing an optimal action, we assume that the probabilities of the outcomes y and z are p and q.

Strategy of player 2:

- action a_2 if the outcome is an element of $\{x, y\}$
- action b_2 if the outcome is z.

Strategy of player 1:

- If the outcome is x then an action which is optimal if player 2 performs the action a_2
- outcome $\in \{y, z\}$:
  - with probability $\frac{p}{p+q}$ an action which is optimal if player 2 performs the action a_2
  - with probability $\frac{q}{p+q}$ an action which is optimal if player 2 performs the action b_2.

$\rightsquigarrow$

A correlated equilibrium of a strategic game $\langle N, (A_i), (u_i) \rangle$ consists of

1) a finite probability space (Ω, π) where Ω is the set of outcomes and π is a probability distribution on Ω,

2) for each player $i \in N$ a partition P_i of Ω the socalled <u>information partition</u> of player i and

3) for each player $i \in N$ a function

$$\sigma_i : \Omega \to A_i$$

with $\sigma_i(\omega) = \sigma_i(\omega')$ if ω and ω' are contained in the same set of the partition P_i,

Such that $\forall\, i \in N$ and all $\tilde{\sigma}_i : \Omega \to A_i$ with $\tilde{\sigma}_i(\omega) = \tilde{\sigma}_i(\omega')$ if ω and ω' are contained in the same set of P_i hold

$$\sum_{\omega \in \Omega} \pi(\omega) \cdot u_i(\sigma_{-i}(\omega), \sigma_i(\omega))$$

$$(*) \qquad \geq \sum_{\omega \in \Omega} \pi(\omega)\, u_i(\sigma_{-i}(\omega), \tilde{\sigma}_i(\omega))$$

Goal:

Proof that the set of correlated equilibria contains the set of mixed Nash equilibria.

<u>Theorem 5.1</u>

Let $\Gamma = \langle N, (A_i), (u_i)\rangle$ be a finite strategic game. Let $x = (x_1, x_2, \ldots, x_n) \in X$ be a mixed Nash equilibrium of Γ. Then there exists a correlated equilibrium $\langle (\Omega, \pi), (P_i), (\sigma_i)\rangle$

in which for each player $i \in N$ the distribution on A_i induced by σ_i is x_i.

<u>Proof</u>:

Let

$$\Omega := A = A_1 \times A_2 \times \dots \times A_n \ , \ \text{where } n = |N|$$

and π be defined by

$$\pi(a) := \prod_{j=1}^{n} x_j(a_j) \quad \text{for} \quad a = (a_1, a_2, \dots, a_n) \in A$$

For each $i \in N$ and $a_j^i \in A_i$ let

$$Q_i(a_j^i) = \{ a \in A \mid a_i = a_j^i \}.$$

The information partition P_i of the player i, $i \in N$ consists of the m_i sets $Q_i(a_j^i)$.

Then we define $\sigma_i : \Omega \to A_i$ by

$$\sigma_i(a) = a_i \quad \forall \ a = (a_1, a_2, \dots, a_n) \in \Omega.$$

Definition of a mixed Nash equilibrium implies

$$\sum_{a \in \Omega} \pi(a) \cdot u_i(\sigma_{-i}(a), \sigma_i(a))$$

$$\geq \sum_{a \in \Omega} \pi(a) \ u_i(\sigma_{-i}(a), \tau_i(a))$$

Note that the right side of the inequality corresponds to the special case that player i

chooses the action $J_i(a)$ with probability $x_i(a)$.

$\Rightarrow$

$\langle (\Omega, \pi), (P_i), (\sigma_i) \rangle$ is a correlated Nash equilibrium where the probability distribution induced by π and σ_i is equal to x_i.

∎

The following example shows that the other direction is not valid. This means that there is a correlated equilibrium which has no corresponding mixed Nash equilibrium.

Example:

We consider the game BoS again. Remember that BoS has two pure Nash equilibria

$$(B,B) \quad \text{and} \quad (S,S).$$

The payoff profiles of these two Nash equilibria are

$$(2,1) \quad \text{and} \quad (1,2).$$

Question:

What are the payoff profiles of the other Nash equilibria?

To answer this question we consider any mixed Nash equilibrium (x_1, x_2).

- If $x_1(B) = 1$ or $x_1(B) = 0$ then we obtain the payoff profiles $(2,1)$ and $(1,2)$ again.
- Assume that $0 < x_1(B) < 1$.

Because of the observation after the proof of Nash's theorem (p. 101) we obtain

$$x_1(B) > 0 \Rightarrow U_1(x_{-1}, \pi_{1,B}) = U_1(x)$$

Hence,

$$1 \cdot x_2(B) \cdot 2 = x_1(B)\, x_2(B) \cdot 2 + x_1(S) \cdot x_2(S) \cdot 1$$

$$\Leftrightarrow \quad \underbrace{(1 - x_1(B))}_{x_1(S)} \cdot 2\, x_2(B) = x_1(S) \cdot x_2(S)$$

$$\Leftrightarrow \quad 2\, x_2(B) = x_2(S)$$

This implies

$$x_2(B) = \tfrac{1}{3} \quad \text{and} \quad x_2(S) = \tfrac{2}{3}.$$

Analogously, we derive from $x_1(S) > 0$

$$x_1(B) = 2\, x_1(S).$$

This implies

$$x_1(B) = \tfrac{2}{3} \quad \text{and} \quad x_1(S) = \tfrac{1}{3}$$

Altogether, we obtain the following additional mixed Nash equilibrium.

$$\left(\left(\tfrac{2}{3}, \tfrac{1}{3}\right), \left(\tfrac{1}{3}, \tfrac{2}{3}\right)\right)$$

- The cases $x_2(B) = 1$, $x_2(B) = 0$ and $0 < x_2(B) < 1$ lead to the same mixed Nash equilibria.

$\Longrightarrow$

Exactly the following payoff profiles of a mixed Nash equilibrium are possible.

$$(2,1), \ (1,2) \text{ and } \left(\tfrac{2}{3}, \tfrac{2}{3}\right).$$

Goal:

To prove that a correlated equilibrium with another payoff profile exists.

Consider

$$\Omega = \{\text{head}, \text{number}\}$$
$$\pi(\text{head}) = \pi(\text{number}) = \tfrac{1}{2}$$
$$P_1 = P_2 = [\{\text{head}\}, \{\text{number}\}] \text{ and}$$
$$\sigma_i(\text{head}) = B, \ \sigma_i(\text{number}) = S \text{ for } i = 1,2$$

This is a correlated equilibrium with payoff profile $\left(\tfrac{3}{2}, \tfrac{3}{2}\right)$

$\Longrightarrow$

There exists a correlated equilibrium which has no corresponding Nash equilibrium.

Exercise:

Which are the three correlated equilibria which correspond to the three mixed Nash equilibria of BoS?

6. The price of anarchy for randomized load balance games

Given

- $S = \{s_1, s_2, \dots, s_m\}$ m servers
- $J = \{t_1, t_2, \dots, t_n\}$ n jobs

  $|t_j| = w_j$ size or weight of job t_j

  $S_j \subseteq S$ the subset of servers which can execute the job t_j

Independently to the other jobs, job t_j chooses a probability distribution

$$p^j = (p_1^j, p_2^j, \dots, p_m^j)$$

on S which is consistent to S_j; i.e.,

$$s_k \notin S_j \Rightarrow p_k^j = 0.$$

- Each server s_k has a load dependent responce time $r_k : \mathbb{Q}^+ \to \mathbb{Q}^+$

(First, we shall consider only the special case $r_k(x) = x \;\; \forall k \in \{1, 2, \dots, m\}$.)

we have to compute a set $\{ p^j \mid 1 \leq j \leq n \}$ of probability distributions.

Goal:

The definition of a Nash equilibrium where each job is selfish and wish to optimize his own utility.

expected load of the server s_k:

$$L_k = \sum_{j=1}^{n} p_k^j \cdot w_j$$

First we shall define the expected load c_j of that server which executes the job t_j

1. guess:

$$c_j := \sum_{i=1}^{m} p_i^j L_i$$

Note that this approach does not realize the fact that at the moment when a job decides its server its whole weight is assigned to that server.

Example:

$$J = \{t_1\}, \quad w_1 = 1, \quad p_k^1 = \tfrac{1}{m}, \quad 1 \leq k \leq m$$

For $1 \leq k \leq m$, we obtain

$$L_k = p_k^1 \cdot w_1 = \frac{1}{m} \cdot 1 = \frac{1}{m}$$

and applying the definition above

$$c_1 = \sum_{k=1}^{m} p_k^1 \cdot L_k = \frac{1}{m}.$$

But the correct value of c_1 has to be 1!

2. guess

Better is the following definition:

$$c_j := \sum_{i=1}^{m} p_i^j \left(w_j + \sum_{k \neq j} p_i^k w_k \right)$$

$$= \sum_{i=1}^{m} p_i^j \underbrace{\left(L_i + (1 - p_i^j) w_j \right)}$$

$\hat{=}$ expected load of the server s_i under the condition that s_i executes the job t_j.

A profile is a mixed Nash equilibrium iff for each job t_j and each server s_i with $p_i^j > 0$, the expected load of s_i under the condition that s_i executes t_j is not larger than the expected load of any other machine s_k with $s_k \in S_j$ under the condition that s_k executes the job t_j. More formally, we obtain