8. Markets and market equilibria

8.1 A simple market game

Given

- n sellers of a product (for example ice cream)
- scenario

  The buyers are equally distributed on the line segment $[0,1] \subset \mathbb{R}$. Each of them buys the ice cream at a nearest seller. The buyers are equally distributed to the nearest sellers.

The intention of each seller is to choose his location on $[0,1]$ such that his gain is maximized.

We shall discuss the Nash equilibria for the cases $n \in \{1,2,3\}$.

$n = 1$:

The only seller has a monopoly. Hence, he can choose any point on $[0,1]$ to obtain a Nash equilibrium.

$n = 2$:

If the shops of both sellers are not placed at the same point each seller can improve his gain if he places his shop closer to the shop of the other player.

$\Rightarrow$

In a Nash equilibrium, both shops are placed at the same point.

If the shops are placed at a point $\neq 0{,}5$ then each player can enlarge his gain by placing his shop closer to $0{,}5$.

$\Rightarrow$

The only Nash equilibrium is both shops placed at the point $0{,}5$.

<u>$n = 3$:</u>

In dependence on the placements of the shops, we distinguish three cases.

<u>Case 1:</u> The three shops are not placed at the same point.

We consider a shop which is

- the only shop on his point and
- which is not placed between the other two shops.

Since not all three shops are placed at the same place, such a shop exists.

This shop can improve his gain by choosing a point which is closer to the other two shops.

$\Rightarrow$

This is not a Nash equilibrium.

Case 2: The three shops are placed at the same point $\neq 0{,}5$.

Then each seller can enlarge his gain if he places his shop at the point 0,5. Instead of a third of the buyers, he would serve at least the half of the buyers.

$\Rightarrow$

This is not a Nash equilibrium.

Case 3: The three shops are placed at the point 0,5.

Then each of the three sellers can enlarge his gain if he places his shop a little bit away from the point 0,5. Then instead of a third of the buyers he would serve almost the half of the buyers.

$\Rightarrow$

This is not a Nash equilibrium.

Altogether, we have shown that in the case $n = 3$ no pure Nash equilibrium exists.

Exercise:

Does there exist a mixed Nash equilibrium in the case $n = 3$? What is the situation for $n > 3$?

Implicitely, we have assumed that the prices of ice cream are the same for all sellers.

Assume that each seller can determine his own price for his ice cream. Then among other things, the following problems arise:

- Maybe, for the buyers there is a tradeoff between the price of the product and the closeness of the shop.
- The quantity bought by a customer can de= pend on the price of the product.

What is the social value of a solution of the game above? Possibly, we can consider the ratio

$$\frac{\text{quantity}}{\text{price}}.$$

$\Rightarrow$

A seller has to find a balance between the maxi= mation of the profit and the minimization of the price.

$\rightsquigarrow$

Our goal is the development of a model for a market game with prices.

8.2 A market game with prices.

Given

- n markets $M = \{m_1, m_2, \dots, m_n\}$
- k sellers $A = \{a_1, a_2, \dots, a_k\}$
- L_j, $1 \le j \le k$ set of possible locations of the seller a_j.
- bipartite weighted graph $G = (M, L, E, c)$ such that

$$L := \bigcup_{j=1}^{k} L_j,$$

$$E := \{ (i, e) \mid 1 \le i \le n,\ e \in L \}$$

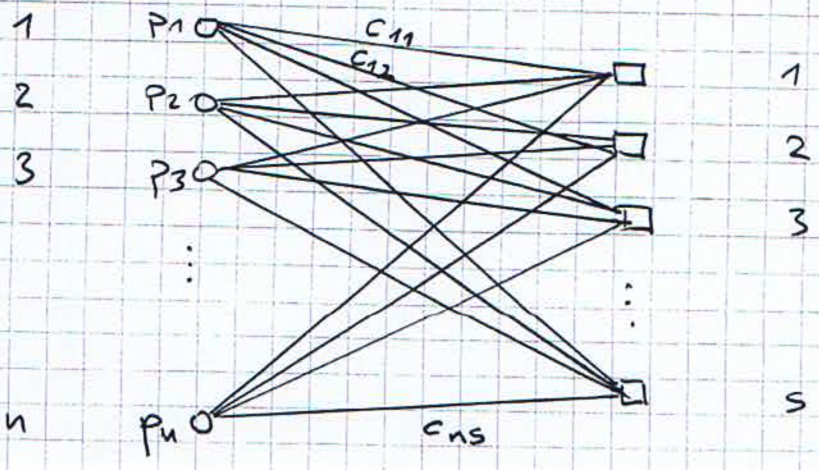
c_{ie}, $(i,e) \in E$ cost for serving the market m_i from location e.

- p_i, $1 \le i \le n$ the maximal price which can be paid by market m_i.

Let $L = \{ l_1, l_2, \dots, l_s \}$. Then the edge set E can be defined as follows:

$$E := \{ (i, j) \mid 1 \le i \le n,\ 1 \le j \le s \}$$

$\rightsquigarrow$ markets



If a market m_r cannot be served from a location ℓ_t then this can be modeled by setting $c_{rt} := \infty$.

For the special case $c_{ij} \neq c_{is}$ for $j \neq s$, we shall prove that the market game with prices is a potential game.

Notations:

Let $L' \subseteq L$ be the set of locations which have a seller.

π_{ij}, $1 \leq i \leq n$ denotes the price of the seller at location $\ell_j \in L'$ for the market m_i.

Analysis:

(a) From the point of view of the markets

Each market m_i buys his goods from the cheapest seller. Let $\sigma(i)$ denote the location of the cheapest seller for the market m_i if such a seller exists; i.e.,

$$\sigma(i) = \begin{cases} j_0 \text{ where } \ell_{j_0} \in L' \text{ and} & \\ \pi_{ij_0} = \min\limits_{\ell_k \in L'} \pi_{ik} \leq p_i & \text{if } j_0 \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases}$$

b) From the point of view of the sellers

The seller at location $\ell_j \in L'$ attempts to hold his price π_{ij} with respect to the market m_i below the prices of the other sellers. This means that π_{ij} should be below π_{ik}, $k \neq j$, $\ell_k \in L'$. Furthermore, his gain should be positive; i.e.,

$$\pi_{ij} > c_{ij}.$$

$\Rightarrow$

The seller at location $\ell_j \in L'$ can offer a price below all other sellers iff for all $\ell_k \in L' \setminus \{\ell_j\}$ there holds

$$c_{ij} < c_{ik}.$$

$\Rightarrow$

$$\sigma(i) = \begin{cases} j_0 \text{ where } \ell_{j_0} \in L' \text{ and} & \\ \quad c_{ij_0} = \min\limits_{\ell_k \in L'} c_{ik} < p_i & \text{if } j_0 \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Determination of the prices

In dependence if

$$\min \{ c_{ik} \mid \ell_k \in L' \setminus \{\ell_{\sigma(i)}\} \} \leq p_i$$

or not, the seller at location $\sigma(i)$

chooses the second smallest value c_{ik} with $l_k \in L'$ or p_i.

$\rightsquigarrow$

$$\pi_{i\sigma(i)} := \begin{cases} \min\{p_i, \min\{c_{ik} \mid l_k \in L' \setminus \{l_{\sigma(i)}\}\}\} & \text{if } c_{i\sigma(i)} \leq p_i \\ \text{undefined} & \text{otherwise} \end{cases}$$

Theorem 8.1

If for all i for all j, k with $j \neq k$, $c_{ij} < \infty$ and $c_{ik} < \infty$ also $c_{ij} \neq c_{ik}$ then the market game with prices is a potential game.

Proof:

Let $M' \subseteq M$ be the set of markets m_i such that there is a seller which offers a price $\leq p_i$. We define the potential function ϕ in the following way.

$$\phi := \sum_{m_i \in M'} c_{i\sigma(i)} + \sum_{m_i \notin M'} p_i.$$

We have to prove that ϕ is indeed a potential function.

Definition of potential function $\Rightarrow$

For doing this it suffices to show the following:

If a single seller changes his location then the difference of the values of ϕ is equal to the change of the profit of the sellers which changes his location.

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Assume that the seller a_c changes from location l_j to location l_k.

Let

ϕ be the potential function before the change

ϕ' be the potential function after the change

$\sigma'(i)$ be the location of the cheapest seller for the market m_i after the change, $1 \le i \le n$.

<u>Agreement:</u>

If $m_i \notin M'$ then $c_{i\sigma(i)}$ and $c_{i\sigma'(i)}$, respectively denotes the price p_i.

We partition the set M of markets into the following four pairwise disjoint sets:

$$S_1 := \{ i \mid \sigma(i) = j \text{ and } \sigma'(i) = k \}$$

$$S_2 := \{ i \mid \sigma(i) = j \text{ and } \sigma'(i) \neq k \}$$

$$S_3 := \{ i \mid \sigma(i) \neq j \text{ and } \sigma'(i) = k \}$$

$$S_4 := \{ i \mid \sigma(i) \neq j \text{ and } \sigma'(i) \neq k \}.$$

P_i, $1 \le i \le 4$ denotes the change of the profit of the seller a_c with respect to the markets in S_i.

- For each market $i \in S_1$ there holds:
  - Before the change of the location:

    $l_k \notin L'$ and π_{ij} does not depend on l_j.
  - After the change of the location

    $l_j \notin L'$ and π_{ik} does not depend on l_k.

  Hence, it follows from the definition of the prices:

$$\pi_{ik} = \pi_{ij}.$$

  Therefore, we obtain the following change of the profit with respect to the market m_i:

$$(\pi_{ik} - c_{i\sigma'(i)}) - (\pi_{ij} - c_{i\sigma(i)})$$

$$= c_{i\sigma(i)} - c_{i\sigma'(i)}.$$

  Hence,

  - $P_1 = \sum_{i \in S_1} (c_{i\sigma(i)} - c_{i\sigma'(i)}).$

- For each market $i \in S_2$ there holds:

$$\pi_{ij} = c_{i\sigma'(i)}.$$

  This follows from the fact that the location corresponding to the second smallest cost is the location assigned to the market m_i after the change of location.

Hence, the profit of the seller a_c is reduced by

$$-(\pi_{ij} - c_{i\sigma(i)})$$
$$= -(c_{i\sigma'(i)} - c_{i\sigma(i)})$$
$$= c_{i\sigma(i)} - c_{i\sigma'(i)}.$$

Therefore, we obtain

- $P_2 = \sum_{i \in S_2} (c_{i\sigma(i)} - c_{i\sigma'(i)})$.

Analogously, we obtain

- $P_3 = \sum_{i \in S_3} (c_{i\sigma(i)} - c_{i\sigma'(i)})$ and
- $P_4 = \sum_{i \in S_4} (c_{i\sigma(i)} - c_{i\sigma'(i)})$.

<u>Exercise:</u>

Show that P_3 and P_4 above are given correctly.

In total, we obtain the following change of profit of the seller a_c:

$$P = P_1 + P_2 + P_3 + P_4$$
$$= \sum_{i \in M} (c_{i\sigma(i)} - c_{i\sigma'(i)})$$
$$= \left(\sum_{i \in M'} c_{i\sigma(i)} + \sum_{i \notin M'} p_i\right)$$
$$- \left(\sum_{i \in M''} c_{i\sigma'(i)} + \sum_{i \notin M''} p_i\right)$$

where M'' is the set of markets m_i after the change of the location such that there is a seller offering a price $\leq p_i$.

This proves that Φ is indeed a potential function. ☒

Theorem 1.5 $\Rightarrow$

<u>Corollary 8.1</u>

The special case above of the market game with prices has a pure Nash equilibrium.

The social value of a solution is given by the sum of the contributions of all markets. The contribution of a market which is not served by a seller is zero. The contribution of a market m_i served by the seller at location $\sigma(i)$ consists of

- the saving $p_i - \pi_{i\sigma(i)}$ of the market m_i and
- the profit $\pi_{i\sigma(i)} - c_{i\sigma(i)}$ of the seller

Hence, the total contribution is

$$p_i - c_{i\sigma(i)}$$

Therefore, the social value of a solution is

$$\sum_{i=1}^{n} p_i - \Phi.$$

<u>Question</u>:

How relates the social value of a pure Nash equilibrium and the social value of an optimal solution?

The following theorem gives an answer to this question.

Theorem 8.2

The social value of an optimal solution of the special case above of the market game with prices is at most the double of the social value of a pure Nash equilibrium.

Proof:

Given any pure Nash equilibrium let

- $j_1, j_2, \dots, j_k$ be the locations of the sellers in the Nash equilibrium
- $j_1', j_2', \dots, j_k'$ be the locations of the sellers in the optimal solution
- $\sigma(i)$ the location of the seller assigned to market m_i in the Nash equilibrium
- $\sigma'(i)$ the location of the seller assigned to market m_i in the optimal solution

We identify seller and the index of his location.

Consider the possible change of seller ℓ from location j_ℓ to location j_ℓ'.

Theorem 8.1 $\Longrightarrow$

improvement of the utility of seller ℓ $\hat{=}$ exactly the improvement of the potential function ϕ (15)

$\Longrightarrow$

This is exactly the improvement of the social value

$$\left(\sum_{i=1}^{n} p_i\right) - \phi .$$

The solution is a pure Nash equilibrium

$\Longrightarrow$

The change of a single seller does not improve his profit and hence, also not the social value.

But the change of the location of ≥ 2 sellers could improve the social value.

Goal:

Estimation of the best possible improvement because of such a change of locations.

Let

- $val'(\ell)$ the new profit of the seller ℓ if he change his location alone from j_ℓ to j'_ℓ.
- $\delta(i)$ the difference between the contributions to the social value of the market m_i in the optimal solution and in the Nash equilibrium.