Can we get an implementation of an entire phase by performing something like BFS followed by something like DFS?

1.2.2 Nonbipartite graphs

Let $G=(V,E)$ be an undirected graph, $M \subseteq E$ be a matching of G, and $G_M=(V',E_M)$ be the directed graph as defined in Section 1.1.1.

Goal:

Construction from G_M a layered and directed graph $\overline{G}_M=(V',\overline{E}_M)$ such that

1. the l-th layer contains exactly those nodes $[v,X]\in V'$ such that a shortest strongly simple path from s to $[v,X]$ in G_M has length l, and
2. $\overline{G}_M$ contains all shortest strongly simple paths from s to t in G_M.

It is clear that s is the only node in Layer 0, i.e., $\text{level}(s)=0$.

Structure of G_M $\Longrightarrow$

$X=B$ $(X=A)$ $\Rightarrow$ $\text{level}([v,X])$ is odd (even).

~~BFS on G_M with start node s finds shortest~~ simple distances from s, and not shortest strongly simple distances.

Idea:

Modify BFS such that the modified breadth-first search (MBFS) finds strongly simple distances

MBFS is seperated into phases. During Phase ℓ, the ℓ-th level of $\bar{G}_M = (V', \bar{E}_M)$ is constructed.

Remember that for the construction of the $(\ell+1)$st level, BFS needs only to consider the nodes in Level ℓ, and to insert into the $(\ell+1)$st level all nodes w which fulfill the following properties.

1. There is a node v in the ℓth level with $(v,w) \in E$.
2. Level(w) has not been defined.

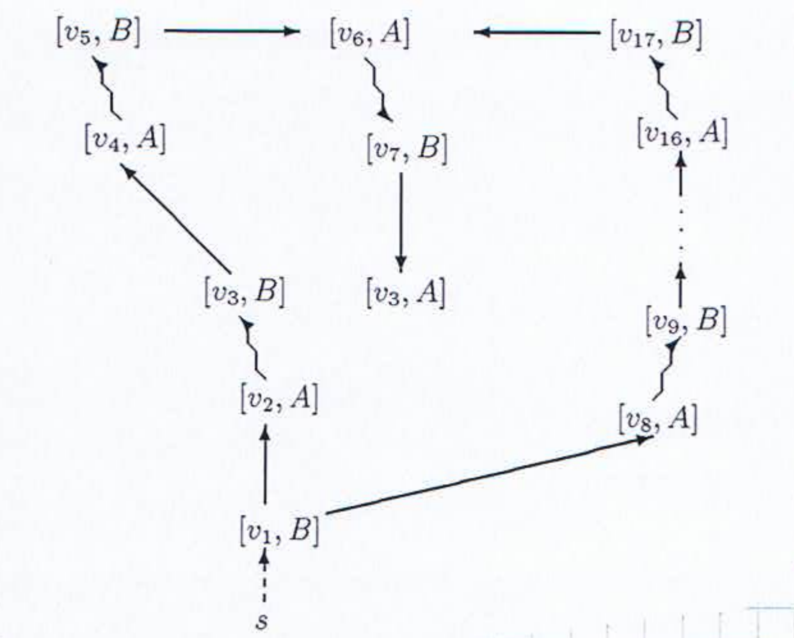
With respect to finding strongly simple distances from s, the construction of the $(\ell+1)$st level is a bit more difficult. By the structure of G_M, the level of a non-free node $[w,B]$ is well-defined by the level of the unique node $[v,A]$ with $([v,A],[w,B]) \in E_M$. Hence, the construction of odd levels is trivial.

For odd ℓ we shall describe the construction of the $(\ell+1)$st level under the assumption that Levels $0,1,2,\ldots,\ell$ are constructed.

It is clear that similar to BFS, MBFS has to insert into the $(\ell+1)$st level all nodes $[w,A] \in V'$ which fulfill the following properties:

1. There is a node $[v,B]$ in Level ℓ with $([v,B],[w,A]) \in E_M$, and there is a strongly simple path from s to $[v,B]$ of length ℓ which does not contain $[w,B]$.
2. Level$([w,A])$ has not been defined.

But these are not all nodes which MBFS has to insert into Level $\ell+1$. Consider the following example.



Note that level$([v_7,B]) = 7$, but level$([v_3,A]) \neq$
since the unique shortest strongly simple path from s to $[v_7,B]$ contains $[v_3,B]$.

The unique strongly simple path P from s to $[v_3,A]$ has length 14. Hence, level$([v_3,A]) = 14$. Furthermore, level$([v_7,A]) = 4$, level$([v_{17},A]) = 6$ and so on. Moreover,

$$level([v_3, A]) = (level([v_{17}, B]) + level([v_6, B]) + 1) - level([v_3, B]).$$

$\Rightarrow$

MBFS has to insert nodes $[w, A] \in V'$ into level $l+1$ for which there is a shortest strongly simple path $P = s, [v_1, B], [v_2, A], \ldots, [v_l, B], [w, A]$ with $level([v_l, B]) < l$.

Similary to breadth-first search MBFS considers after the construction of the l-th level, l odd, all edges $([v, B], [w, A])$ with $level([v, B]) = l$. We distinguish three cases

Case 1: Level$([w, A]) > l$ and there is a strongly simple path P from s to $[v, B]$ of length l not containing $[w, B]$.

MBFS inserts node $[w, A]$ into the $(l+1)$-st level, and adds the edge $([v, B], [w, A])$ to $\bar{E}_M$.

Case 2: Level$([w, A]) > l$, and all strongly simple paths from s to $[v, B]$ of length l contain $[w, B]$.

MBFS does not enlarge Level $l+1$.

Case 3: Level$([w, A]) \leq l$.

Level$([w, A])$ is already defined. But later, possibly the edge $([v, B], [w, A])$ is contained in a shortest strongly simple path $P = P', [x,B], [u,A]$ from s to $[u,A]$ where level$([x,B]) <$ level$([u,A]) - 1$.

Note that these are all cases. For the correct treatment of Case 1 and of Case 2, MBFS has to know whether there exists a shortest strongly simple path P from s to $[v,B]$ which does not contain the node $[w,B]$. For getting this knowledge, the following notation is useful.

For all nodes $[v,X] \in V'$ such that level$([v,X])$ is defined, we denote by DOM$([v,X])$ the node $[u,B] \in V'$ which satisfies:

a) All shortest strongly simple paths from s to $[v,X]$ contain $[u,B]$.
b) Level$([u,A])$ has not been defined.
c) Level$([w,B]) \leq$ level$([u,B])$ for all $[w,B] \in V'$ satisfying a) and b).

If such a node $[u,B]$ does not exist then DOM$([v,X])$ denotes the node s. Furthermore, DOM(s) denotes node s. Note the fact that level$([u,B])$ is defined but level$([u,A])$ is not defined implies DOM$([u,B]) = \{[u,B]\}$

Next, we shall show that always $|\text{DOM}([v,X])| = 1$. This will be a direct consequence of the following lemma.

Lemma 1.5

Let $P = s, [v_1, B], [v_2, A], \ldots, [v_\ell, X]$ be a shortest strongly simple path from s to $[v_\ell, X]$; i.e., $\text{level}([v_\ell, X]) = \ell$. Let $[v_j, B] \in P$ be a node with $\text{level}([v_j, A]) > \ell$. Then the following hold true:

a) $\text{Level}([v_j, B]) = j$.

b) $\text{Level}([v_i, B]) < \text{level}([v_j, B])$ for all odd $i < j$.

Proof:

a) Assume $\text{level}([v_j, B]) < j$.

Let $Q, [v_j, B]$ be any shortest strongly simple path from s to $[v_j, B]$. Let

$$R := Q, [v_j, B], \underbrace{[v_{j+1}, A], \ldots, [v_\ell, X]}_{P_2}.$$

Since $|R| < |P|$ and P is a shortest strongly simple path from s to $[v_\ell, X]$, the paths Q and P_2 cannot be strongly disjoint.

Let $[z, Y]$ be the first node on Q such that $[z, Y]$ on P_2 or $[z, \overline{Y}]$ on P_2. Let

$Q = Q_1, [z, Y], Q_2$ and

$$P_2 = \begin{cases} P_{21}, [z, Y], P_{22} & \text{if } [z, Y] \text{ on } P_2 \\ P_{21}, [z, \overline{Y}], P_{22} & \text{if } [z, \overline{Y}] \text{ on } P_2 \end{cases}$$

If $[z, Y]$ on P_2 then

$$S = Q_1, [z, Y], P_{22}$$

would be a strongly simple path from s to $[v_e, X]$ shorter than $level([v_e, X])$, a contradiction.

Hence, $[z, \bar{Y}]$ is on P_2. But then

$$S = Q_1, [z, Y], r(P_{21}), [v_j, A]$$

would be a strongly simple path from s to $[v_j, A]$ shorter than ℓ, a contradiction.

b)

Exercise

The proof of Lemma 1.5 represents the main technique used in the subsequence.

We consider a path from s to a node $[y, X]$ of length $<$ $level([y, X])$. Hence, the path is not strongly simple. The path is constructed such that it separates into two strongly simple parts. Then we consider the first node $[z, Y]$ on the first part such that $[z, Y]$ or $[z, \bar{Y}]$ is on the second part. If on the second part there is the same node $[z, Y]$ as on the first part then we easily obtain a shorter strongly simple path from s to $[y, X]$ by concatenation of the sub-path from s to $[z, Y]$ of the first part and the subpath directly after $[z, Y]$ to $[y, x]$ of the second part. Hence, the same node on the second part as on the first part is noncritical.

The node $[z, \bar{Y}]$ is the critical. To get a contradiction in that case is often much more involved. In the subsequence, we shall only give an explicit discussion of the case that the critical node is on the second part.

<u>Lemma 1.6</u>

Let $[v, X] \in V'$ such that level$([v,X])$ is defined. Then the following hold true:

a) $|\mathrm{DOM}([v,X])| = 1$.

b) Let $\mathrm{DOM}([v,X]) = [u,B]$. Then after the definition of level$([u,A])$, always $\mathrm{DOM}([v,X]) = \mathrm{DOM}([u,B])$.

<u>Proof:</u>

a) Assume that $|\mathrm{DOM}([v,X])| > 1$. Let $[u_1,B], [u_2,B] \in \mathrm{DOM}([v,X])$ be two distinct elements of $\mathrm{DOM}([v,X])$. By the consideration of any shortest strongly simple path from s to $[v,X]$ and applying Lemma 1.5, we obtain:

$$\text{level}([u_1,B]) < \text{level}([u_2,B]) \quad \text{or}$$
$$\text{level}([u_2,B]) < \text{level}([u_1,B]) \quad , \text{ a contradiction}$$

b) is obvious by the definition of $\mathrm{DOM}([v,X])$. ∎

The following lemma enables a simple treatment of Case 1 and of Case 2.

Lemma 1.7

Assume MBFS considers the edge $([v,B],[w,A])$ and Case 2 is fulfilled. Then on all shortest strongly simple paths from s to $[v,B]$ there is no node $[z,B]$ between $[w,B]$ and $[v,B]$ with level$([z,A])$ is not defined.

Remark:

Note that Lemma 1.7 implies $\mathrm{DOM}([v,B]) = [w,B]$.

Proof:

Assume that there exists a shortest strongly simple path

$$P = P_1, [w,B], P_2, [z,B], P_3, [v,B]$$

from s to $[v,B]$ with level$([z,A])$ is not defined. Consider the path

$$R = P_1, [w,B], [v,A], r(P_3), [z,A].$$

By construction, R is a strongly simple path from s to $[z,A]$ and $|R| <$ level$([v,B])$. But this contradicts that level$([z,A])$ is not defined.

$\blacksquare$

With respect to Case 3 of MBFS, we have to characterize the shortest strongly simple paths $P = P', [x,B], [u,A]$ from s to a node $[u,A]$ with level$([x,B]) <$ level$([u,A]) - 1$.

Theorem 1.7

Let $P = P', [x,B], [u,A]$ be a shortest strongly simple path from s to $[u,A]$ with level$([x,B]) <$ level$([u,A]) - 1$. Let $[v,B]$ be the last node on $P', [x,B]$ such that the length of the sub=path from s to $[v,B]$ of P is equal level$([v,B])$. Let $P' = P_1, [v,B], [w,A], P_2$. Then at the be=ginning of Phase level$([u,A])$ the following holds true:

DOM$([y,X])$ = DOM$([y,\bar{X}])$ = $[u,B]$ for all nodes $[y,X]$ on $[w,A], P_2, [x,B]$.

Remark:

Since level$([d,B]) = 1$ for the direct successor $[d,B]$ of s on P, the node $[v,B]$ on $P', [x,B]$ exists. Since level$([x,B]) <$ level$([u,A]) - 1$, the node $[v,B]$ is on P'.

Proof:

The proof separates into the proof of four lem=mas. The first lemma shows that for all $[y,X]$ on $[w,A], P_2, [x,B]$ each shortest strongly simple path from s to $[y,X]$ contains the node $[u,B]$. The second lemma shows that on all shortest strongly simple paths from s to $[y,X]$, between $[u,B]$ and $[y,X]$, there is no node $[z,B]$ with level$([z,A]) \geq$ level$([u,A])$. These two lemmas imply directly that