Theorem 1.7

Let $P = P', [x,B], [u,A]$ be a shortest strongly simple path from s to $[u,A]$ with $level([x,B]) < level([u,A]) - 1$. Let $[v,B]$ be the last node on $P', [x,B]$ such that the length of the subpath from s to $[v,B]$ of P is equal $level([v,B])$. Let $P' = P_1, [v,B], [w,A], P_2$. Then at the beginning of Phase $level([u,A])$ the following holds true:

$DOM([y,X]) = DOM([y,\bar{X}]) = [u,B]$ for all nodes $[y,X]$ on $[w,A], P_2, [x,B]$.

Remark:

Since $level([d,B]) = 1$ for the direct successor $[d,B]$ of s on P, the node $[v,B]$ on $P', [x,B]$ exists. Since $level([x,B]) < level([u,A]) - 1$, the node $[v,B]$ is on P'

Proof:

The proof seperates into the proof of four lemmas. The first lemma shows that for all $[y,X]$ on $[w,A], P_2, [x,B]$ each shortest strongly simple path from s to $[y,X]$ contains the node $[u,B]$. The second lemma shows that on all shortest strongly simple paths from s to $[y,X]$, between $[u,B]$ and $[y,X]$, there is no node $[z,B]$ with $level([z,A]) \geq level([u,A])$. These two lemmas imply directly that

$\mathrm{DOM}([y,X]) = [u,B]$ for all $[y,X]$ on $[w,A], P_2, [x,B]$.

The next two lemmas prove the same with respect to $[y,\bar{X}]$ showing $\mathrm{DOM}([y,\bar{X}]) = [u,B]$ for all $[y,X]$ on $[w,A], P_2, [x,B]$.

Lemma 1.8

At the beginning of Phase level$([u,A])$, the following holds true: For all nodes $[y,X]$ on $[w,A], P_2, [x,B]$ for all shortest strongly simple paths $Q, [y,X]$ from s to $[y,X]$, the node $[u,B]$ is on $Q, [y,X]$.

Proof:

Lemma 1.7 $\Rightarrow$ $\mathrm{DOM}([x,B]) = [u,B]$.

Hence, the assertion holds with respect to the node $[x,B]$. Assume that the assertion does not hold with respect to at least one node on $[w,A], P_2$.

Let $[y,X]$ be the last node on $[w,A], P_2$ such that there is a shortest strongly simple path

$$Q, [y,X] \quad \text{from } s \text{ to } [y,X]$$

which does not contain the node $[u,B]$.

We use the paths $Q, [y,X]$ and $[w,A], P_2, [x,B]$ for the construction of a path S from s to $[u,A]$ which is shorter than level$([u,A])$ such that the path S cannot be strongly simple.

Let

$$P_2 = P_{21}, [y,X], P_{22}$$

and

$$S = Q, [y,X], P_{22}, [x,B], [u,A].$$

Construction $\Rightarrow$

The path $P_1, [v,B], [w,A], P_{21}, [y,X]$ is not a <u>shortest</u> strongly simple path from s to $[y,X]$. Hence,

$$|Q,[y,X]| < |P_1,[v,B],[w,A],P_{21},[y,X]|.$$

$\Rightarrow$

$$|S| < |P| = \mathrm{level}([u,A]).$$

Hence, the subpaths Q and $P_{22},[x,B]$ have to be not strongly disjoint.

Let $[z,Y]$ be the last node on $P_{22},[x,B]$ such that $[z,Y]$ or $[z,\bar{Y}]$ is on Q. Let

$$P_{22},[x,B] = P'_{22}, [z,Y], P''_{22} \quad \text{and}$$

$$Q = \begin{cases} Q_1, [z,Y], Q_2 & \text{if } [z,Y] \text{ is on } Q \\ Q_1, [z,\bar{Y}], Q_2 & \text{if } [z,\bar{Y}] \text{ is on } Q. \end{cases}$$

If $[z,Y]$ is on Q then the path

$$S' = Q_1, [z,Y], P''_{22}, [u,A]$$

would be a strongly simple path from s to $[u,A]$ shorter than $|S|$. This contradicts $|P| = \mathrm{level}([u,A])$.

Since we have obtained this contradiction in a trivial way, we call the node $[z, Y]$ <u>noncritical for Q</u>.

If $[z, \overline{Y}]$ is on Q, the construction of a contradiction is much more involved. Hence we call the node $[z, \overline{Y}]$ <u>critical for Q</u>.

Now we investigate the case that the critical node $[z, \overline{Y}]$ is on Q. Let

$$R = R_1, [u, B], R_2, [z, \overline{Y}]$$

be a shortest strongly simple path from s to $[z, \overline{Y}]$. Note that by construction, the node $[u, B]$ has to be on all shortest strongly simple paths from s to $[z, \overline{Y}]$.

Consider the path

$$T = R_1, [u, B], r(P''_{22})$$

We distinguish two cases.

<u>Case 1</u>: $|Q_1, [z, \overline{Y}]| > |T, [z, \overline{Y}]|$

Consider the path

$$U = T, [z, \overline{Y}], Q_2, [y, x].$$

Since $|U| < |Q, [y, x]|$, the path U cannot be strongly simple.

We distinguish two subcases:

<u>Case 1.1</u> R_1 and P''_{22} are strongly disjoint.

Then R_1 and Q_2 have to be not strongly disjoint.

Let $[c,D]$ be the first node on R_1 such that $[c,D]$ or $[c,\bar{D}]$ is on Q_2. Let

$$R_1 = R_{11}, [c,D], R_{12} \quad \text{and}$$

$$Q_2 = \begin{cases} Q_{21}, [c,D], Q_{22} & \text{if } [c,D] \text{ is on } Q_2 \\ Q_{21}, [c,\bar{D}], Q_{22} & \text{if } [c,\bar{D}] \text{ is on } Q_2. \end{cases}$$

If $[c,D]$ is on Q_2 then

$$R_{11}, [c,D], Q_{22}, [y,X]$$

would be a strongly simple path from s to $[y,X]$ shorter than $|U|$, a contradiction.

Hence, $[c,\bar{D}]$ is on Q_2. Consider the path

$$N = R_{11}, [c,D], r(Q_{21}), [z,Y], P''_{22}, [u,A]$$

<u>Exercise</u>. Prove $|N| < \text{level}([u,A])$.

Construction $\Rightarrow$ N is strongly simple

This contradicts $|P| = \text{level}([u,A])$.

<u>Case 1.2</u> R_1 and P''_{22} are not strongly disjoint.

Let $[d, H]$ be the first node on R_1 such that $[d, H]$ or $[d, \bar{H}]$ is on P''_{22}. Let

$$R_1 = R_{11}, [d, H]\, R_{12} \quad \text{and}$$

$$P''_{22} = \begin{cases} P''_{221}, [d, H], P''_{222} & \text{if } [d,H] \text{ is on } P''_{22} \\ P''_{221}, [d, \bar{H}], P''_{222} & \text{if } [d, \bar{H}] \text{ is on } P''_{22} \end{cases}$$

If $[d, H]$ is on P''_{22} then

$$R_{11}, [d, H], P''_{222}, [u, A]$$

would be a strongly simple path from s to $[u, A]$ shorter than level $([u, A])$, a contradiction.

Hence, $[d, \bar{H}]$ is on P''_{22}. Consider the path

$$T' = R_{11}, [d, H], r(P''_{221})$$

Construction $\Rightarrow$ T' is strongly simple

Consider the path

$$U' = T', [z, \bar{Y}], Q_2, [y, X].$$

Since $|U'| < |U|$, the path U' cannot be strongly simple. Hence, R_{11} and Q_2 have to be not strongly disjoint.

Let $[c, D]$ be the first node on R_{11} such that $[c, D]$ or $[c, \bar{D}]$ is on Q_2. Let

$R_{11} = R_{111}, [c,D], R_{112}$ and

$$Q_2 = \begin{cases} Q_{21}, [c,D], Q_{22} & \text{if } [c,D] \text{ is on } Q_2 \\ Q_{21}, [c,\bar{D}], Q_{22} & \text{if } [c,\bar{D}] \text{ is on } Q_2 \end{cases}$$

If $[c,D]$ is on Q_2 then

$$R_{111}, [c,D], Q_{22}, [y,x]$$

would be a strongly simple path from s to $[y,x]$ shorter than $|U'|$, a contradiction.

Hence, $[c,\bar{D}]$ is on Q_2. Consider the path

$$N = R_{111}, [c,D], \sigma(Q_{21}), [z,\bar{y}], P''_{22}, [u,A]$$

<u>Exercise</u> Prove $|N| < \text{level}([u,A])$.

Construction $\Rightarrow$ N is strongly simple.

This contradicts $|P| = \text{level}([u,A])$.

<u>Case 2</u>: $|Q_1, [z,\bar{y}]| \leq |T, [z,\bar{y}]|$.

Consider the path

$$U = Q_1, [z,\bar{y}], \sigma(R_2), [u,A].$$

<u>Exercise:</u> Prove $|U| < \text{level}([u,A])$.

Hence, U cannot be strongly simple such that the subpaths Q_1 and $\sigma(R_2)$ have to be not strongly disjoint.

Let $[c,D]$ be the first node on Q_1 such that $[c,D]$ or $[c,\bar{D}]$ is on $r(R_2)$. Assume that the critical node $[c,\bar{D}]$ is on $r(R_2)$; i.e., $[c,D]$ is on R_2. Let

$$Q_1 = Q_{11}, [c,D], Q_{12} \quad \text{and} \quad R_2 = R_{21}, [c,D], R_{22}.$$

Consider the path

$$W = Q_{11}, [c,D], R_{22}, [z,Y].$$

Construction $\Rightarrow$

Q_{11} and R_{22} are strongly disjoint such that the path W is strongly simple.

If $|Q_{11}| \leq |R_1, [u,B], R_{21}|$ then W would be a strongly simple path from s to $[z,Y]$ which does not contain the node $[u,B]$ of length $\leq$ level$([z,Y])$, a contradiction.

Hence,

$$|Q_{11}| > |R_1, [u,B], R_{21}|.$$

Consider the path

$$W = R_1, [u,B], R_{21}, [c,D], Q_{12}, [z,\bar{Y}], Q_2, [y,X]$$

Since $|W| <$ level$([y,X])$, the path W cannot be strongly simple. Hence, $R_1, [u,B], R_{21}$ and $Q_{12}, [z,\bar{Y}], Q_2, [y,X]$ have to be not strongly disjoint.

Let $[f,K]$ be the first node on $R_1, [u,B], R_{21}$ such that $[f,K]$ or $[f,\bar{K}]$ is on $Q_{12}, [z,\bar{Y}], Q_2, [y,X]$.
Assume that the critical node $[f,\bar{K}]$ is on $Q_{12}, [z,\bar{Y}], Q_2, [y,X]$. Let

$$Q_{12}, [z,\bar{Y}], Q_2, [y,X] = Q', [f,\bar{K}], Q''.$$

In dependence if $[f,K]$ is on R_{21} or on R_1, we distinguish two cases.

Case 2.1: $[f,K]$ is on R_{21}.

Let

$$R_{21} = R_{211}, [f,K], R_{212}.$$

Consider the path

$$N = Q_{11}, [c,D], Q', [f,\bar{K}], r(R_{211}), [u,A].$$

Exercise: Prove $|N| < \text{level}([u,A])$.

Choice of $[c,D]$ $\Rightarrow$

$Q_{11}, [c,D]$ and $r(R_{211})$ are strongly disjoint.

Choice of $[f,K]$ $\Rightarrow$

Q' and $r(R_{211})$ are strongly disjoint.

Altogether, N would be a strongly simple path from s to $[u,A]$ shorter than $\text{level}([u,A])$, a contradiction.

Case 2.2: $[f,K]$ is on R_1.

Then we have the following situation

$$\begin{tikzcd}
{[y,X]} \arrow[r, "P_{22}'"] & {[z,Y]} \arrow[r, "P_{22}''"] & {[x,B],[u,A]} \\
& {[c,D]} \arrow[u, "R_{22}"'] & \\
{[f,\bar{K}]} \arrow[uu, "Q''"] & & \\
{[c,D]} \arrow[u, "Q'"] & {[u,B]} \arrow[uu, "R_{21}"'] & \\
& {[f,K]} \arrow[u, "R_{12}"'] & \\
s \arrow[uu, "Q_{11}"] & s \arrow[u, "R_{11}"'] &
\end{tikzcd}$$

where $R_1 = R_{11}, [f,K], R_{12}$.

Consider the path

$$N_1 = R_{11}, [f,K], r(Q'), [c,\bar{D}], r(R_{21}), [u,A]$$

Exercise: Prove $|N_1| < \text{level}([u,A])$.

Since $|N_1| < \text{level}([u,A])$, the path N_1 cannot be strongly simple.

Choice of $[f,K]$ $\Rightarrow$

R_{11} and $r(Q')$ are strongly disjoint

$\Longrightarrow$

$r(Q')$ and $r(R_{21})$ have to be not strongly disjoint.

Construction of Type 1 on $r(Q')$ and $r(R_{21})$

Let $[g_1, J_1]$ be the first node on $r(Q')$ such that $[g_1, J_1]$ or $[g_1, \bar{J}_1]$ is on $r(R_{21})$. Assume that the critical node $[g_1, \bar{J}_1]$ is on $r(R_{21})$; i.e., $[g_1, J_1]$ is on R_{21}. Let

$$Q' = Q'_1, [g_1, \bar{J}_1], Q'_2 \quad \text{and} \quad R_{21} = R_{211}, [g_1, J_1], R_{212}$$

Consider the path

$$N_2 = Q_{11}, [c, D], Q'_1, [g_1, \bar{J}_1], r(R_{211}), [u, A]$$

<u>Exercise</u>: Prove $|N_2| < \text{level}([u, A])$.

Since $|N_2| < \text{level}([u, A])$, the path N_2 cannot be strongly simple.

Choice of $[c, D]$ $\Rightarrow$

$Q_{11}, [c, D]$ and $r(R_{211})$ are strongly disjoint.

Hence, Q'_1 and $r(R_{211})$ have to be not strongly disjoint.

Let $[g_2, J_2]$ be the first node on Q'_1 such that $[g_2, J_2]$ or $[g_2, \bar{J}_2]$ is on $r(R_{211})$. Assume that the critical node $[g_2, \bar{J}_2]$ is on $r(R_{211})$; i.e., $[g_2, J_2]$ is on R_{211}. Let

$$Q'_1 = Q'_{11}, [g_2, J_2], Q'_{12} \quad \text{and}$$

$$R_{211} = R_{2111}, [g_2, J_2], R_{2112}.$$

Consider the path

Construction of Type 2 on Q_1' and $r(R_{211})$

$$N_3 = R_{11}, [f,K], r(Q_2'), [g_1, J_1], r(Q_{12}'), [g_2, J_2], r(R_{2111}), [u,A]$$

<u>Exercise</u>: Prove $|N_3| < \mathrm{level}([u,A])$.

Since $|N_3| < \mathrm{level}([u,A])$, the path N_3 cannot be strongly simple.

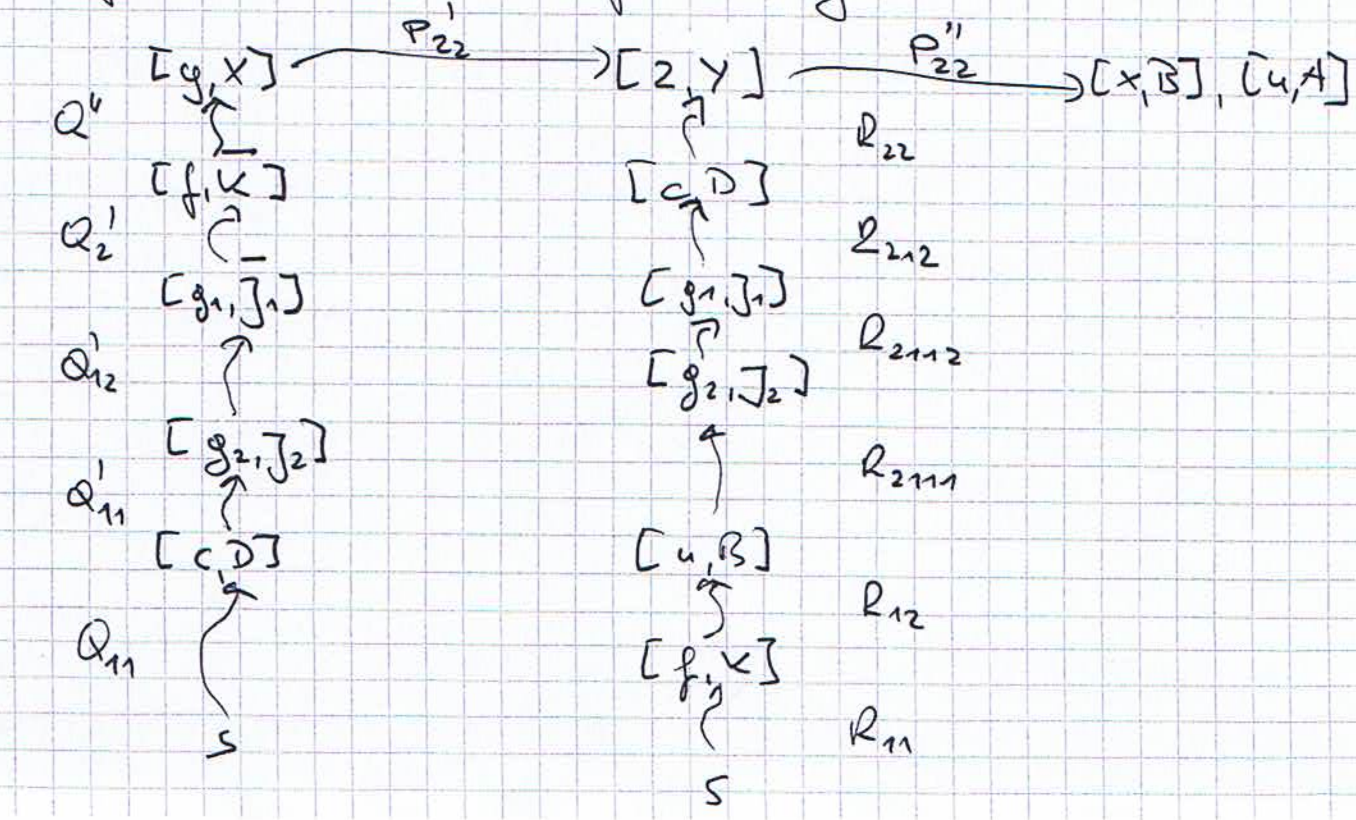
As observed above, R_{11} and $r(Q')$ are strongly disjoint.

$\Longrightarrow$ R_{11} and $r(Q_2'), [g_1, J_1], r(Q_{12}')$ are strongly disjoint.

Choice of $[g_1, J_1]$ $\Longrightarrow$

$r(Q_2')$ and $r(R_{2111})$ are strongly disjoint.

Hence, $r(Q_{12}')$ and $r(R_{2111})$ have to be not strongly disjoint. We have the following situation:



Now we perform a construction of Type 1 on $r(Q'_{12})$ and $r(R_{2111})$. Then we perform a construction of Type 2 on the resulting sub-paths, and so on.

After each such a construction, the resulting subpaths of Q' and R_{21} are proper subpaths of the previous subpaths. Hence, after a finite number of such constructions, the considered path N_j has to be strongly simple such that a contradiction is obtained.

This proves the lemma.

■

<u>Lemma 1.9</u>

At the beginning of Phase level($[u,A]$), the following holds true: For all $[y,x]$ on $[w,A], P_2, [x,B$ on all shortest strongly simple paths from s to $[y,x]$, between $[u,B]$ and $[y,x]$ no node $[z,B]$ with level$([z,A]) \geq$ level$([u,A])$ exists.

<u>Proof</u>:

Assume that the assertion does not hold. Let $[y,x]$ be the last node on $[w,A], P_2, [x,B]$ such that there is a shortest strongly simple path

$$Q = Q_1, [u,B], Q_2, [z,B], Q_3\ [y,x]$$