from s to $[y, X]$ with level$([z, A]) \geq$ level$([u, A])$.

Let

$$[w, A], P_2, [x, B] = P_{21}, [y, X], P_{22}.$$

We use the paths Q and $[w, A], P_2, [x, B]$ for the construction of a path S from s to $[z, A]$, which is shorter than level$([z, A])$.

Consider the path

$$S = Q_1, [u, B], r(P_{22}), [y, \bar{X}], r(Q_3), [z, A]$$

<u>Exercise:</u> Prove $|S| <$ level$([z, A])$.

Since $|S| <$ level$([z, A])$, the path S cannot be strongly simple.

<u>Claim:</u>

The paths Q_1 and P_{22} are strongly disjoint.

<u>Proof:</u>

exercise. ∎

Claim ⟹

$r(P_{22})$ and $r(Q_3)$ have to be not strongly disjoint.

Let $[c, D]$ be the first node on $r(P_{22})$ such that $[c, D]$ or $[c, \bar{D}]$ is on $r(Q_3)$. Assume that the critical node $[c, \bar{D}]$ is on $r(Q_3)$; i.e., $[c, D]$ is on Q_3. Let

$P_{22} = P_{22}', [c,\bar{D}], P_{22}''$ and $Q_3 = Q_{31}, [c,D], Q_{32}$.

Construction $\Rightarrow$

$$Q_1, [u,B], r(P_{22}''), [c,D], Q_{32}, [y,x]$$

is a strongly simple path from s to $[y,x]$.

Since Q is a shortest strongly simple path from s to $[y,x]$, the following holds true:

$$(*) \quad |Q_2, [z,B], Q_{31}| \leq |P_{22}''|.$$

Let

$$R = R_1, [u,B], R_2, [c,\bar{D}]$$

be a shortest strongly simple path from s to $[c,\bar{D}]$. Note that by Lemma 1.5

$$|R_1, [u,B]| = |Q_1, [u,B]| = \text{level}([u,B]).$$

Exercise:

Prove that $Q_1, [u,B], R_2, [c,\bar{D}]$ is a shortest strongly simple path from s to $[c,\bar{D}]$.

Consider the path

$$T = Q_1, [u,B], R_2, [c,\bar{D}], r(Q_{31}), [z,A].$$

$$(*) \Rightarrow |T| < \text{level}([u,A]).$$

Hence, T cannot be strongly simple. Note that by the exercise above, Q_1 and R_2 are strongly disjoint

Hence, R_2 and $r(Q_{31})$ have to be not strongly disjoint.

Let $[a,C]$ be the first node on R_2 such that $[a,C]$ or $[a,\bar{C}]$ is on $r(Q_{31})$. Assume that the critical $[a,\bar{C}]$ is on $r(Q_{31})$; i.e., $[a,C]$ is on Q_{31}. Let

$$R_2 = R_{21}, [a,C], R_{22} \quad \text{and} \quad Q_{31} = Q_{31}', [a,C], Q_{31}''.$$

Consider the path

$$U = Q_1, [u,B], R_{21}, [a,C], Q_{31}'', [c,D], Q_{32}, [y,X].$$

We distinguish two cases:

<u>Case 1</u>: $|R_{21}| < |Q_2, [2,B], Q_{31}'|$

Then $|U| < |Q|$.

Since Q is a shortest strongly simple path from s to $[y,X]$, the path U cannot be strongly simple. Hence, R_{21} and Q_{32} have to be not strongly disjoint.

Note that by the choice of $[a,C]$, the paths R_{21} and Q_{31}'' are strongly disjoint.

Let $[b,H]$ be the first node on R_{21} such that $[b,H]$ or $[b,\bar{H}]$ is on Q_{32}. Assume that the critical node $[b,\bar{H}]$ is on Q_{32}. Let

$$R_{21} = R_{21}', [b,H], R_{21}'' \quad \text{and} \quad Q_{32} = Q_{32}', [b,\bar{H}], Q_{32}''.$$

Consider the path

$L = Q_1, [u,B], R'_{21}, [b,H], r(Q'_{32}), [c,\bar{D}], r(Q_{31}), [z,A]$

By construction, L is a strongly simple path from s to $[z,A]$.

Moreover, $(*)$ implies that $|R'_{21}| < |P''_{22}|$ and hence, $|L| <$ level$([u,A])$.

This contradicts level$([z,A]) \geq$ level$([u,A])$. Altogether, we have proved that Case 1 cannot happen.

Case 2: $|R_{21}| \geq |Q_2, [z,B], Q'_{31}|$.

Consider the path

$L = Q_1, [u,B]\; Q_2, [z,B], Q'_{31}, [a,C], R_{22}, [c,\bar{D}]$

Since $[z,B] \in L$, the choice of the node $[y,X]$ implies that L cannot be a shortest strongly simple path from s to $[c,\bar{D}]$.

Since $|L| \leq |R|$, the subpaths

$Q_2, [z,B], Q'_{31}$ and R_{22}

have to be not strongly disjoint.

Let $[f,K]$ be the first node on $Q_2, [z,B], Q'_{31}$ such that $[f,K]$ or $[f,\bar{K}]$ is on R_{22}. Assume that the critical node $[f,\bar{K}]$ is on R_{22}.
Let

$$R_{22} = R'_{22}, [f, \bar{k}], R''_{22}.$$

In dependence if $[f,k]$ is on Q'_{31} or on Q_2, we distinguish two subcases. Note that $[2,B]$ and also $[2,A]$ cannot be on R_{22}.

Case 2.1 $[f,k]$ is on Q'_{31}.

Let

$$Q'_{31} = Q'_{311}, [f,k], Q'_{312}.$$

Consider the path

$$N = Q_1, [u,B], R_{21}, [a,C], R'_{22}, [f,\bar{k}], r(Q'_{311}), [2,A]$$

Exercise: Prove $|N| <$ level$([u,A])$.

Choice of $[a,C]$ $\Rightarrow$

$R_{21}, [a,C]$ and $r(Q'_{311})$ are strongly disjoint

Choice of $[f,k]$ $\Rightarrow$

R'_{22} and $r(Q'_{311})$ are strongly disjoint.

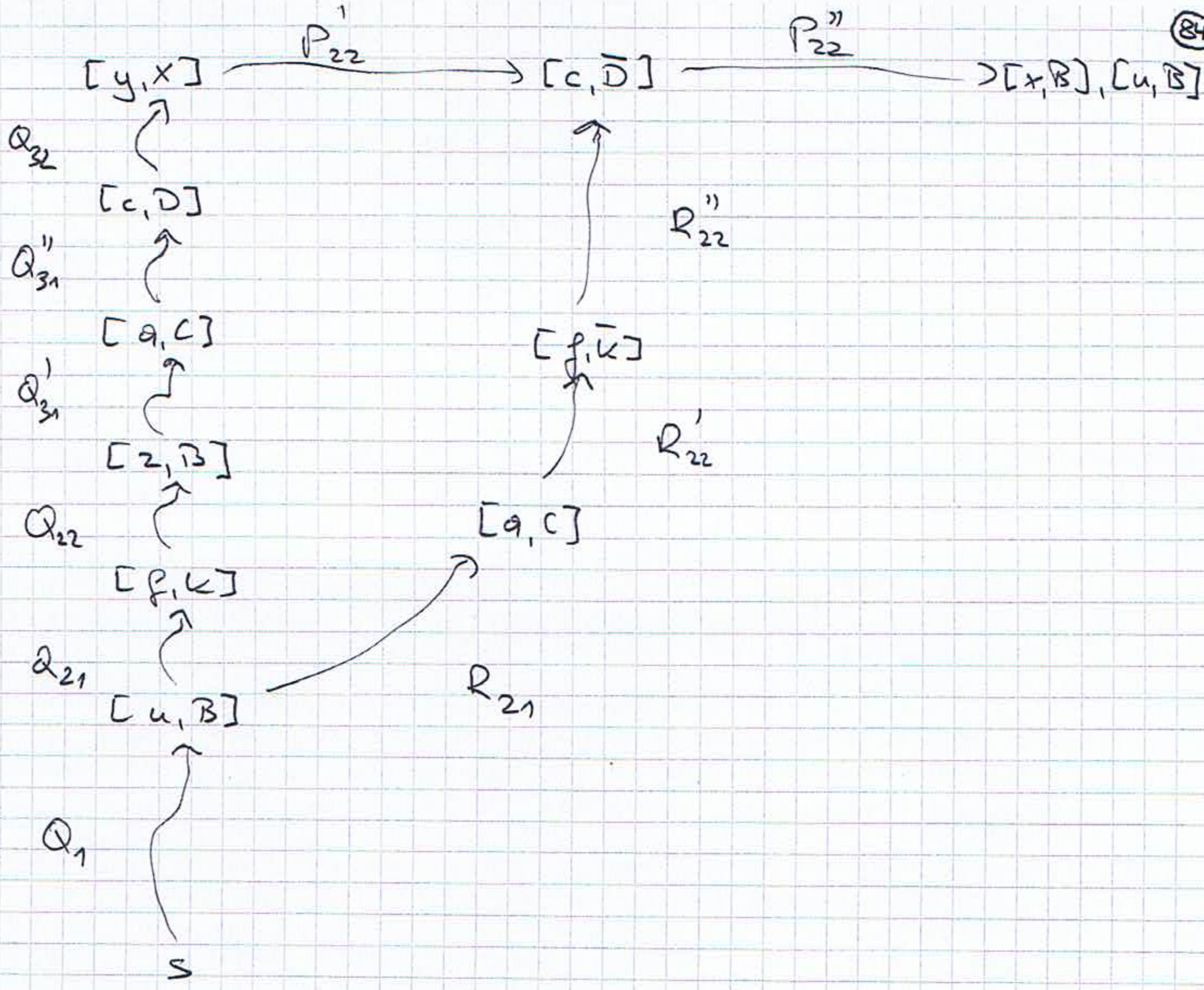
Altogether, N would be a strongly simple path from s to $[2,A]$ shorter than level$([u,A])$, a contradiction.

Case 2.2 $[f,k]$ is on Q_2.

Let

$$Q_2 = Q_{21}, [f,k], Q_{22}.$$

Then we have the following situation:



The proof is analogous to the proof of Lemma 1.8, Case 2.2, where

[z, B] plays the rôle of [u, B],
R'_{22} plays the rôle of Q', and
Q'_{31} plays the rôle of R_{21}.

<u>Exercise</u>: Work out the proof of Subcase 2.2.

This proves the lemma.

∎

Note that Lemma 1.8 and Lemma 1.9 imply $DOM([y,X]) = [u,B]$ for all $[y,X]$ on $[w,A], P_2, [x,B]$. The next two lemmas prove that $DOM([y,\bar{X}]) = [u,B]$ for all $[y,X]$ on $[w,A], P_2, [x,B]$.

Lemma 1.10

At the beginning of Phase level$([u,A])$, the following holds true: For all $[y,X]$ on $[w,A], P_2, [x,B]$ for all shortest strongly simple paths $Q, [y,\bar{X}]$ from s to $[y,\bar{X}]$, the node $[u,B]$ is on $Q, [y,\bar{X}]$.

Proof:

Assume that the assertion does not hold with respect to some nodes on $[w,A], P_2, [x,B]$.

Let $[y,X]$ be the last node on $[w,A], P_2, [x,B]$ such that there is a shortest strongly simple path

$$Q, [y,\bar{X}] \quad \text{from s to } [y,\bar{X}]$$

which does not contain the node $[u,B]$. Let

$$[w,A], P_2, [x,B] = P_{21}, [y,X], P_{22}$$

Let

$$R = R_1, [u,B], R_2, [y,X]$$

be any shortest strongly simple path from s to $[y,X]$. Consider the path

$T = Q, [y, \bar{x}], r(R_2), [u, A]$

<u>Exercise</u>: Prove $|T| < level([u,A])$

The rest of the proof is the same as the proof of Lemma 1.8.

∎

<u>Lemma 1.11</u>

At the beginning of Phase $level([u,A])$, the following holds true: For all $[y,x]$ on $[u,A], P_2, [x,B]$ on all shortest strongly simple paths from s to $[y,\bar{x}]$, between $[u,B]$ and $[y,\bar{x}]$ no node $[z,B]$ with $level([z,A]) \geq level([u,A])$ exists.

<u>Proof</u>:

Analogous to the proof of Lemma 1.9.

Exercise.

∎

This finishs the proof of Theorem 1.7

∎

Theorem 1.7 gives a characterization of the shortest strongly simple paths $P = P', [x,B], [u,A]$ from s to a node $[u,A]$ with $level([x,B]) < level([u,A]) - 1$. As we can see in the proof of the theorem, such a path is the concatenation of a shortest strongly simple path P_1 from s to a node $[v,B]$ with $[u,B] \notin P_1$ and a backpath $[w,A], r(Q_2), [u,A]$ where

$$Q = Q_1, [u,B], Q_2\ [w,B]$$

is a shortest strongly simple path from s to $[w,B]$.

The following theorem shows that in a situation described by Theorem 1.7 each such a path is a shortest strongly simple path from s to $[u,A]$.

Theorem 1.8

Let $[v,B]$, $[w,A]$, $[u,B] \in V'$ such that

i) $([v,B],[w,A]) \in E_M$,

ii) the ℓ-th level is defined, where

$$\ell = level([v,B]) + level([w,B]) - level([u,B]),$$

and

iii) $DOM([v,B]) \neq DOM([w,A]) = DOM([w,B]) = [u,B]$

Then the following is fulfilled.

a) There exists a shortest strongly simple path P_1 from s to $[v,B]$ with $[u,B] \notin P_1$.

b) Let P_1 be any shortest strongly simple path from s to $[v,B]$ with $[u,B] \notin P_1$. Let

$$Q = Q_1, [u,B], Q_2, [w,B]$$

be any shortest strongly simple path from s to $[w,B]$. Then

$$P = P_1, [w,A], r(Q_2), [u,A]$$

is a shortest strongly simple path from s to $[u,A]$.

Proof:

a) Assume that $[u,B] \in R$ for all shortest strongly simple paths R from s to $[v,B]$.

Let $[z,B] = DOM([v,B])$. Let R be a shortest strongly simple path from s to $[v,B]$.

Lemma 1.5 $\Rightarrow$

$$R = R_1, [u,B], R_2, [z,B], R_3, [v,B].$$

Let $Q = Q_1, [u,B], Q_2, [w,B]$ be a shortest strongly simple path from s to $[w,B]$ with $[z,B] \notin Q$. Since $DOM([w,B]) = [u,B]$, the path Q exists.

Consider the path

$$S = Q_1, [u,B], Q_2, [w,B], [v,A], r(R_3), [z,A]$$

The rest is analogous to the proof of Lemma 1.9, where $l+1$ plays the rôle of level$([u,A])$.

<u>Exercise:</u> Work out the proof of Theorem 1.8 a).

b) It suffices to show that P is strongly simple. Note that $|P| = l+1$.

Assume that P is not strongly simple; i.e., P_1 and $r(Q_2)$ are not strongly disjoint.

Let $[q,X]$ be the first node on P_1 such that $[q,X]$ or $[q,\bar{X}]$ is on $r(Q_2)$. Let

$$P_1 = P_{11}, [q,X], P_{12} \quad \text{and}$$

$$Q_2 = \begin{cases} Q_{21}, [q,X], Q_{22} & \text{if } [q,X] \text{ is on } Q_2 \\ Q_{21}, [q,\bar{X}], Q_{22} & \text{if } [q,\bar{X}] \text{ is on } Q_2. \end{cases}$$

If $[q,\bar{X}] \in Q_2$ then

$U = P_{11}, [q,X], r(Q_{21}), [u,A]$

would be a strongly simple path from s to $[u,A]$.

Furthermore, $|U| < |P| = \ell+1$.

This contradicts $level([u,A]) > \ell$.

Hence, $[q,X] \in Q_2$. We distinguish two cases.

Case 1: $|P_{11}| \leq |Q_{11}, [u,B], Q_{21}|$.

Consider the path

$T = P_{11}, [q,X], Q_{22}, [w,B]$.

Choice of $[q,X]$ $\Rightarrow$ T is strongly simple.

Furthermore, $|T| \leq |Q|$.

Since Q is a shortest strongly simple path, it follows $|T| = |Q|$.

But then, T is a shortest strongly simple path from s to $[w,B]$ which does not contain $[u,B]$.

This contradicts $DOM([w,B]) = [u,B]$.

Case 2: $|P_{11}| > |Q_{11}, [u,B], Q_{21}|$.

Consider the path

$T = Q_{11}, [u,B], Q_{21}, [q,X], P_{12}$.

Since $|T| < |P_1|$, the path T cannot be strongly simple. Hence, $Q_1, [u,B], Q_{21}$ and P_{12} have to be not strongly disjoint.

Let $[y,C]$ be the first node on $Q_1, [u,B], Q_{21}$ such that $[y,C]$ or $[y,\bar{C}]$ is on P_{12}. Let

$$P_{12} = \begin{cases} P_{121}, [y,C], P_{122} & \text{if } [y,C] \text{ is on } P_{12} \\ P_{121}, [y,\bar{C}], P_{122} & \text{if } [y,\bar{C}] \text{ is on } P_{12}. \end{cases}$$

and

$$Q_1, [u,B], Q_{21} = Q', [y,C], Q''.$$

If $[y,C] \in P_{12}$ then

$$U = Q', [y,C], P_{122}$$

would be a strongly simple path from s to $[v,B]$ shorter than P_1, a contradiction.

Hence, $[y,\bar{C}] \in P_{12}$.

In dependence if $[y,C]$ is on Q_{21} or on Q_1, we distinguish two subcases.

<u>Case 2.1</u> $[y,C]$ is on Q_{21}.

Let

$$Q_{21} = Q_{211}, [y,C], Q_{212}.$$

Consider the path

$U = P_{11}, [q, X], P_{121}, [y, \overline{C}], r(Q_{211}), [u, A].$

By the choice of $[q, X]$ and $[y, C]$, the path U is strongly simple.

Construction $\Rightarrow$

$$|U| < level([v, B]) + level([w, B]) - level([u, B]) + 1$$
$$= \ell + 1.$$

This contradicts $level([u, A]) > \ell$.

Case 2.2 $[y, C]$ is on Q_1.

Let
$$Q_1 = Q_{11}, [y, C], Q_{12}.$$

Consider the path

$$U = Q_{11}, [y, C], r(P_{121}), [q, \overline{X}], r(Q_{21}), [u, A].$$

Since $|P_{11}| > |Q_1, [u, B], Q_{21}|$ we obtain

$|U| < |P|$ and hence, $|U| < \ell$.

Choice of $[y, C]$ $\Rightarrow$

Q_{11} and $r(P_{121})$ are strongly disjoint.

Hence, the paths $r(P_{121})$ and $r(Q_{21})$ have to be not strongly disjoint.

Let $[r, D]$ be the first node on $r(P_{121})$ such that $[r, D]$ or $[r, \overline{D}]$ is on $r(Q_{21})$. Assume that the critical node $[r, \overline{D}]$ is on $r(Q_{21})$; i.e., $[r, D]$ is on Q_{21}. Let

$P_{121} = P'_{121}, [r,\bar{D}], P''_{121}$ and $Q_{21} = Q'_{21}, [r,D], Q''_{21}$.

Consider the path

$$S = P_{11}, [q,X], P'_{121}, [r,\bar{D}], r(Q'_{21}), [u,A].$$

Choice of $[q,X]$ $\Rightarrow$

P_{11} and $r(Q'_{21})$ are strongly disjoint.

Choice of $[r,D]$ $\Rightarrow$

P'_{121} and $r(Q'_{21})$ are strongly disjoint.

Hence, S is a strongly simple path from s to $[u,A]$

Construction $\Rightarrow$

$$|S| < |P_1| + |Q| - level([u,B]) \leq \ell$$

This contradicts $level([u,A]) > \ell$.

Altogether, we have proved Part b) of Theorem 1.8

∎

Now we can finish the description of the algorithm MBFS.

Assume that the ℓ-th level, ℓ odd is constructed. We shall describe Phase $\ell+1$; i.e., the construction of Level $\ell+1$.