Let B be any nonsingular $m \times m$ matrix composed of m linearly independent columns of A; i.e, B is a basis of A. The components of x corresponding to the columns of B are called basic variables; the other components of x are called nonbasic variables with respect to the basis B.

A point $x \in \mathbb{R}^n$ with $Ax = b$ and the property that all nonbasic variables with respect to B are equal to zero is said to be a basic solution with respect to the basis B.

Given a basis B, we obtain after setting the corresponding nonbasic variables to zero the following system of m equations in m unknowns:

$$B x_B = b$$

which is uniquely solvable for the basic variables x_B. If a basic solution x with respect to a basis B in nonnegative then it is called a basic feasible solution.

The following corollary is a direct consequence of Theorem 2.1.

Corollary 2.1

A point $x \in P$ is a vertex of P iff x is a basic feasible solution with respect to some basis B.

There are only $\binom{n}{m}$ possibilities to choose m columns of an $m \times n$ matrix A. Hence, we obtain the following corollary.

Corollary 2.2
The polyhedron P has only a finite number of vertices.

Exercise:
Let P be a polytope. Prove that each point $x \in P$ is a convex combination of the vertices ~~of~~ P.
(Hint: Prove first the assertion for all vertices, then for each point on an edge, then for all points on a facet and finally for each point in the interior of P.)

08.07

A vector $d \in \mathbb{R}^n \setminus \{0\}$ is called a direction of a polyhedron P if for each point $x_0 \in P$, the ray $\{x \in \mathbb{R}^n \mid x = x_0 + \lambda d, \lambda \geq 0\}$ lies entirely in P. Obviously P is unbounded iff P has a direction.

The following lemma characterizes the directions of a polyhedron.

Lemma 2.1
Let $d \neq 0$. Then d is a direction of $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ iff $Ad = 0$ and $d \geq 0$.

Proof:

„$\Rightarrow$"

Suppose that d is a direction of P. Then

$\{x \in \mathbb{R}^n \mid x = x_0 + \lambda d, \lambda \geq 0\} \subseteq P$

for all $x_0 \in P$.

Assume that $d \ngeq 0$.

Then there exists a component d_i of d with $d_i < 0$.

For λ' large enough, for any $x_0 \in P$ there holds

$$x_0^i + \lambda' d_i < 0$$

where x_0^i is the i-th component of x_0.

$\Rightarrow$

$x_0 + \lambda' d \ngeq 0$ and hence, $x_0 + \lambda' d \notin P$.

This contradicts that d is a direction of P.

Therefore

$$d \geq 0.$$

Assume that $Ad \neq 0$.

$\Rightarrow$

$$A\lambda d \neq 0 \quad \text{for } \lambda > 0.$$

This implies for any point $x_0 \in P$

$$A(x_0 + \lambda d) = Ax_0 + A\lambda d = b + A\lambda d \neq b.$$

$\Longrightarrow$

$$x_0 + \lambda d \notin P$$

This contradicts that d is a direction of P.
Therefore

$$Ad = 0.$$

"$\Longleftarrow$"

Suppose that $Ad = 0$ and $d \geq 0$.

$\Longrightarrow$

$$A(x_0 + \lambda d) = Ax_0 + A\lambda d = b + \lambda Ad = b$$

for all $x_0 \in P$ and all $\lambda \geq 0$.

Definition $\Longrightarrow$

d is a direction of P.

∎

Lemma 2.1 implies directly that for each direction $d \in \mathbb{R}^n$ of a polyhedron P and each $\lambda > 0$, the vector λd is also a direction of P.

If $x \in P$ is not a convex combination of the vertices of P then x can be described as the sum of a point which is a convex combination of the vertices and a direction of P. This observation gives us the following simple re= presentation theorem:

Theorem 2.2

Let $P \subseteq \mathbb{R}^n$ be any polyhedron and let

$\{v_i \mid i \in I\}$ be the set of vertices of P. Then every point $x \in P$ can be represented as

$$x = \sum_{i \in I} \lambda_i v_i + d$$

where $\sum_{i \in I} \lambda_i = 1$, $\lambda_i \geq 0$ for all $i \in I$ and either $d = 0$ or d is a direction of P.

Proof:

exercise

∎

Theorem 2.2 implies directly the following corollary.

Corollary 2.3

A nonempty polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ has at least one vertex.

Now we can prove the Fundamental theorem of linear programming.

Theorem 2.3

Let P be a nonempty polyhedron. Then the minimum value of $z(x) = c^T x$ for $x \in P$ is attained at a vertex of P or z has no lower bound on P.

Proof:

If P has a direction d with $c^T d < 0$ then

P is unbounded and the value of z converges on the direction d to $-\infty$.

Otherwise, the minimum is attained at points which can be expressed as convex combinations of the vertices v_i of P. Let

$$\hat{x} = \sum_{i \in I} \lambda_i v_i$$

be any such a point where $\{v_i \mid i \in I\}$ is the set of vertices of P, $\sum_{i \in I} \lambda_i = 1$ and $\lambda_i \geqslant 0$ for all $i \in I$. Then

$$c^T \hat{x} = c^T \sum_{i \in I} \lambda_i v_i = \sum_{i \in I} \lambda_i c^T v_i$$

$$\geqslant \min \{ c^T v_i \mid i \in I \}$$

Hence, the minimum of z is attained at a vertex of P.

■

Theorem 2.3 implies that for getting an optimal solution of a linear programming problem it suffices to consider basic feasible solutions and to investigate if there is a direction along which $z \to -\infty$.

Assume that for a given linear program

$$\min \; z(x) = c^T x$$
$$Ax = b$$
$$x \geq 0$$

there is a finite optimal solution.

Since the number of basic solutions can be $\binom{n}{m}$ where m is the number of rows and n is the number of columns of A, for large m and n we cannot consider all basic solutions for the computation of an optimal feasible basic solution. Hence, we need

1. a strategy for the consideration of the basic solutions and
2. a criterion which decides if the current considered feasible basic solution is optimal.

2.2 The simplex method

Goal:

Development of a method to solve the linear program

$$\min \; z(x) = c^T x$$
$$Ax = b$$
$$x \geq 0$$

where A is an $(m \times n)$-matrix of row rank m. The case $\operatorname{rank}(A) < m$ will be discussed later on.

Geometric motivation

- Start in any vertex x_0 of the polyhedron

$$P = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$$

Go from vertex to vertex along edges of P that are "downhill" with respect to the objective function $z(x) = c^T x$, generating a sequence of vertices with strictly decreasing objective value.

$\Rightarrow$

Once the method leaves a vertex the method can never return to that vertex.

$\Rightarrow$

In a finite number of steps, a vertex will be reached which is optimal, or an edge will be chosen which goes off to infinity and along which z goes to $-\infty$.

Question:

How to convert the above geometric description of the simplex method into an algebraic and, hence, computational form?

The vertex x_0 corresponds to a basic feasible solution

$$x_0 = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}.$$

The corresponding objective value $z(x_0)$ is obtained by

$$z(x_0) = c_B^T B^{-1} b$$

where c_B contains exactly the components of c which correspond to the basic variables x_B.

Goal:

The computation of a socalled downhill edge which starts in x_0 and on which the objective value $z(x)$ strictly decreases.

Two cases are possible:

1. The downhill edge ends in a neighboured vertex x' with $z(x') < z(x_0)$.
2. The edge has an infinite length such that the objective value $z(x)$ converges on the edge to $-\infty$.

In the second case, the algorithm knows that no finite optimum exists and terminates. In the first case, the algoritum looks for a down=hill edge which starts in x'. If no such an edge exists, the algorithm terminates.

To prove the correctness of the method we have to show that a basic feasible solution for which no downhill edge exists is always an optimum solution.

To transform the geometric consideration into an algebraic method, we have to solve the following problems:

1. If $P \neq \emptyset$ then find a vertex of P.
2. Given a vertex of P compute a downhill edge which starts in this vertex if such an edge exists. If no such an edge exists this has to be established.

First we shall investigate the second problem.

Let B be the basis corresponding to a given vertex of the polyhedron P. Let

$$x_0 = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1} b \\ 0 \end{pmatrix}$$

be the corresponding basic feasible solution where $A = [B, N]$ and $c^T = [c_B^T, c_N^T]$ are partitioned with respect to basic and nonbasic variables. $Ax = b$ can be described as

$$B x_B + N x_N = b$$

Since B is nonsingular the inverse matrix B^{-1} exists.

$\Rightarrow$

In dependence to the variables x_N corresponding to the nonbasic variables, the values x_B corresponding to the basic variables can be expressed as follows:

$$(1) \qquad x_B = B^{-1}b - B^{-1}N\,x_N$$

After the elimination of x_B in the equation

$$z(x) = c_B^T x_B + c_N^T x_N$$

we obtain

$$(2) \qquad z(x) = c_B^T B^{-1} b - \left(c_B^T B^{-1} N - c_N^T \right) x_N$$

Two bases are called <u>neighboured</u> if they differ only in one column. A basic solution where at least one basic variable has the value zero is called <u>degenerate</u>. Otherwise, the basic solution is <u>nondegenerate.</u>

Neighboured vertices of the polyhedron correspond to neighboured bases.

$\Longrightarrow$

Going from one vertex to a neighboured vertex is equivalent to the exchange of one column of the corresponding basis by another where each other column of the basis remains to be unchanged.

For simplicity assume that the basic feasible solution x_0 is nondegenerate.

Going from x_0 to a neighboured vertex corresponds to increasing the value of a nonbasic variable x_j where all other nonbasic variables remain to be zero.

Question:

- How to find an appropriate nonbasic variable x_j ?

In other words

- How to find a downhill edge which starts in x_0 ?

For getting an answer to this question let us consider equation (2) again.

Increasing the nonbasic variable x_j and all other nonbasic variables remain to be zero implies that the second summand has the value

$$-(c_B^T B^{-1} a_j - c_j)\, x_j$$

where a_j is the j-th column of A.

$$\bar{c}_j := -(c_B^T B^{-1} a_j - c_j)$$

is called reduced cost for x_j.

$\Longrightarrow$

x_j corresponds to a downhill edge iff $\bar{c}_j < 0$.

It is useful to combine the equations (1) and (2).

$\rightsquigarrow$

$$\begin{bmatrix} z(x) \\ x_B \end{bmatrix} = \begin{bmatrix} c_B^T B^{-1} b \\ B^{-1} b \end{bmatrix} - \begin{bmatrix} c_B^T B^{-1} N - c_N^T \\ B^{-1} N \end{bmatrix} x_N$$

Let R be the set of the indizes of the columns in N and let

$$z(x) = x_{B_0} \quad \text{and} \quad x_B = (x_{B_1}, x_{B_2}, \dots, x_{B_m})^T.$$

For simplification of the notation we define

$$y_0 := \begin{bmatrix} y_{00} \\ y_{10} \\ \vdots \\ y_{m0} \end{bmatrix} := \begin{bmatrix} c_B^T B^{-1} b \\ B^{-1} b \end{bmatrix}$$

and for $j \in R$; i.e., a_j is a column in N,

$$y_j := \begin{bmatrix} y_{0j} \\ y_{1j} \\ \vdots \\ y_{mj} \end{bmatrix} := \begin{bmatrix} c_B^T B^{-1} a_j - c_j \\ B^{-1} a_j \end{bmatrix}.$$

Then we can write (1) and (2) for $i = 0, 1, \dots, m$ as follows

(3) $$x_{B_i} = y_{i0} - \sum_{j \in R} y_{ij} x_j.$$

If we set in (3)

$$x_j = 0 \quad \text{for all } j \in R$$

then we obtain the basic solution which corres= ponds to the basic B.

Definition ⟹

$$y_{0j} = -c_j \quad \text{for all } j \in R.$$

Suppose that x_B is nondegenerate and that

$$y_{0q} > 0 \quad \text{for any } q \in R.$$

Increasing x_q and simultaneous fixing the other nonbasic variables to zero decreases x_{B_0} proportional to y_{0q}.

Moreover, each x_{B_i} is a linear function of x_q and decreased proportional to y_{iq}.

If $y_{iq} > 0$ then

$$x_{B_i} \geq 0 \text{ as long as } x_q < \frac{y_{i0}}{y_{iq}}.$$

At the moment when

$$x_q = \frac{y_{i0}}{y_{iq}}$$

there holds $x_{B_i} = 0$.

If $y_{iq} \leq 0$ for $1 \leq i \leq m$ then x_q can be increased arbitrary without a basic variable

x_{B_i}, $1 \le i \le m$ becomes to be negative such that the solution remains to be feasible.

$\Rightarrow$

We can improve the value of the objective function within the feasible polyhedron arbitrary.

$\Rightarrow$

The given linear program is <u>unbounded</u>.

<u>Exercise:</u>
Show that every unbounded linear program has such a basic feasible solution.

$\rightsquigarrow$

<u>Theorem 2.4</u>
A linear program LP is unbounded iff there is a basic feasible solution x_B and a nonbasic variable x_q such that for the vector y_q corresponding to x_q there hold:

$$y_{0q} > 0 \text{ and } y_{iq} \le 0 \text{ for } 1 \le i \le m.$$

For the development of the simplex method we assume that the given linear program is bounded.

Let x_{B_p} be any basic variable with

$$0 < \frac{y_{p0}}{y_{pq}} = \min_{1 \le i \le m} \left\{ \frac{y_{i0}}{y_{iq}} \;\middle|\; y_{iq} > 0 \right\}$$

If we increase x_q to $\frac{y_{p0}}{y_{pq}}$ and fix the other nonbasic variables to zero then we obtain

$$x_q = \frac{y_{p0}}{y_{pq}}$$

$$x_{B_i} = y_{i0} - y_{iq} \frac{y_{p0}}{y_{pq}} \quad \text{for } i = 0, 1, \ldots, m.$$

We obtain a new basic feasible solution with

$$x_q > 0 \,, \; x_{B_p} = 0 \text{ and } x_{B_0} = y_{00} - y_{0q} \frac{y_{p0}}{y_{pq}}.$$

Since $y_{0q} > 0$ and $\frac{y_{p0}}{y_{pq}} > 0$ the value x_{B_0} decreases strictly.

Nex we shall investigate the computation of the values corresponding to the new basis the new y_{ij}'s in the equations (3).

We obtain the value corresponding to the new basic variable x_q if we solve the equation in (3) which corresponds to x_{B_p} for x_q. After doing this we obtain

(4) $$x_q = \frac{y_{p0}}{y_{pq}} - \sum_{j \in R \setminus \{q\}} \frac{y_{pj}}{y_{pq}} \cdot x_j - \frac{1}{y_{pq}} x_{B_p}.$$

The value which corresponds to the basic variable x_{B_i}, $i \neq p$ is obtained if we eliminate x_q in (3) using (4).

$\rightsquigarrow$

$$x_{B_i} = y_{i0} - \frac{y_{iq} y_{p0}}{y_{pq}} - \sum_{j \in R \setminus \{q\}} \left(y_{ij} - \frac{y_{iq} y_{pj}}{y_{pq}} \right) x_j + \frac{y_{iq}}{y_{pq}} x_{B_p}.$$

Let R' denotes the set of the indezes of the nonbasic variables after the exchange of the basic variable x_{B_p} and the nonbasic variable x_q. Then

$$R' = \{B_p\} \cup R \setminus \{q\}.$$

Let y'_{ij} be the new value for y_{ij} with respect to the equations (3). Then there hold for $0 \leq i \leq m$, $i \neq p$:

$$y'_{i0} = y_{i0} - \frac{y_{iq} y_{p0}}{y_{pq}},$$

$$y'_{ij} = y_{ij} - \frac{y_{iq} y_{pj}}{y_{pq}} \quad \text{for } j \in R' \setminus \{B_p\}$$

and

$$y'_{iB_p} = - \frac{y_{iq}}{y_{pq}}.$$