The old p-th basic variable x_{B_p} has been replaced by the new p-th basic variable x_q. Hence, we obtain from equation (4)

$$y'_{p0} = \frac{y_{p0}}{y_{pq}},$$

$$y'_{pj} = \frac{y_{pj}}{y_{pq}} \quad \text{for } j \in R' \setminus \{B_p\} \text{ and}$$

$$y'_{pB_p} = \frac{1}{y_{pq}}.$$

Goal:

Correctness proof for the Simplex method.

We have to show that a basic feasible solution is optimum if the corresponding vertex of the polyhedron has no downhill edge. This means that

$$y_{0j} \leq 0 \quad \text{for all } j \in R.$$

For doing this let B be any basis with basic feasible solution x' and $y_{0j} \leq 0 \;\; \forall j \in R$.

(2) and (3) $\Longrightarrow$

$$(5) \quad z(x) = c_B^T B^{-1} b - \sum_{j \in R} y_{0j} x_j \quad \forall x \in P.$$

Since $y_{0j} \leq 0$ and $x_j \geq 0 \;\; \forall j \in R$, the value $c_B^T B^{-1} b$ is a lower bound for $z(x)$.

Because of

$$x'_B = B^{-1} b \quad \text{and} \quad x'_N = 0$$

we obtain

$$z(x') = c_B^T B^{-1} b.$$

Hence, x' is an optimum solution.

Altogether, we have proved the following theorem.

<u>Theorem 2.5</u>

The basic solution described by the equations (3) is an optimal solution of the given linear program if the following properties are fulfilled:

1. $y_{i0} \geq 0$ for $i = 1, 2, \dots, m$ (feasibility)
2. $y_{0j} \leq 0$ for all $j \in R$ (optimality)

Now we can present an algorithm for the simplex method. We assume the the computed basic solutions are nondegenerate. Later on we shall extend the algorithm such that the computed basic solution can be degenerate.

We assume that at the beginning, an initial basic feasible solution is known.

16.07.

Algorithm SIMPLEX

Input: linear program

$$\text{LP}: \quad \min z(x) = c^T x$$
$$Ax = b$$
$$x \geq 0.$$

Output: An optimum solution for LP if the optimum is bounded and the information "optimum unbounded" otherwise.

Method:

(1) Initialisation:

Start with a basic feasible solution $(x_B, 0)$.

(2) Optimality test:

If $y_{0j} \leq 0$ for all $j \in R$ then

- the current basic solution $(x_B, 0)$ is an optimum solution. Output $(x_B, 0)$ and STOP.

(3) Determination of a downhill edge:

- Choose an appropriate variable x_q, $q \in R$ to enter the basis.

- Choose a variable x_{B_p} with

$$\frac{y_{p0}}{y_{pq}} = \min_{1 \le i \le m} \left\{ \frac{y_{i0}}{y_{iq}} \;\middle|\; y_{iq} > 0 \right\}$$

to leave the basis if $\frac{y_{p0}}{y_{pq}}$ is defined.

- If $\frac{y_{p0}}{y_{pq}}$ is not defined; i.e., $y_{iq} \le 0$ for $1 \le i \le m$ then LP is unbounded.

  Output "unbounded" and STOP.

(4) Pivot step:

Solve the equations (3) for x_q and x_{B_i}, $i \neq p$ as described above.

$$x_j := 0 \quad \text{for } j \in \{B_p\} \cup R \setminus \{q\}$$

goto (2).

It is possible to implement the simplex method using socalled tables. The following table represent the equations (3).

nonbasic variables

basic variables

		$\cdots$ $-x_j$ $\cdots$	$-x_q$ $\cdots$
x_{B_0}	y_{00}	y_{0j}	y_{0q}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{B_i}	y_{i0}	y_{ij}	y_{iq}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{B_p}	y_{p0}	y_{pj}	y_{pq}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{B_m}	y_{m0}	y_{mj}	y_{mq}

table of type 1.

The i-th row of the table corresponds to the i-th equation. We say that the table is of Type 1.

Next we shall investigate how to perform the change of the basis using tables of Type 1. Suppose that the basic variable x_{B_p} has to be replaced by the nonbasic variable x_q. This can be done as follows:

(1) Divide the p-th row by y_{pq}; ← pivot element

(2) For $0 \le i \le m$, $i \neq p$

- multiply the new p-th row by y_{iq} and subtract the resulting row from the i-th row.

(3) Divide the old q-th column by y_{pq} and multiply this column by -1. In the resulting column replace the component in the p-th row by $\frac{1}{y_{pq}}$. Associate with the new q-th column the new nonbasic variable x_{B_p}.

	·	$\cdots$	$-x_j$	$\cdots$	$-x_{B_p}$	$\cdots$
x_{B_0}	$y_{00} - \left(\frac{y_{0q}y_{p0}}{y_{pq}}\right)$		$y_{0j} - \left(\frac{y_{0q}y_{pj}}{y_{pq}}\right)$		$-\frac{y_{0q}}{y_{pq}}$	
$\vdots$						
x_{B_i}	$y_{i0} - \left(\frac{y_{iq}y_{p0}}{y_{pq}}\right)$		$y_{ij} - \left(\frac{y_{iq}y_{pj}}{y_{pq}}\right)$		$-\frac{y_{iq}}{y_{pq}}$	
$\vdots$						
x_q	$\frac{y_{p0}}{y_{pq}}$		$\frac{y_{pj}}{y_{pq}}$		$\frac{1}{y_{pq}}$	
$\vdots$						

Next we shall investigate how to get the initial feasible basic solution for the given linear program

$$\text{LP}: \quad \min \; z(x) = c^T x$$
$$Ax = b$$
$$x \geq 0.$$

W.l.o.g. we can assume $b \geq 0$. If for an equation

$$a_j^T x = b_j$$

$b_j < 0$ then we can multiply the equation by -1.

<u>Idea</u>:

In dependence of LP, define a linear program LP' such that:

1. LP' has a trivial feasible basic solution such that we can start the algorithm SIMPLEX with LP' and the trivial feasible basic solution.

2. With help of the solution of LP' it is easy to compute a feasible basic solution of LP.

$\leadsto$

Consider

$$x = (x_1, x_2, \dots, x_{n+m})^T,$$

$$\tilde{x} = (x_1, x_2, \dots, x_n)^T, \text{ and}$$

$$\bar{x} = (x_{n+1}, x_{n+2}, \dots, x_{n+m})^T$$

and the following linear program:

$$LP': \quad \min \; z(x) = \sum_{i=n+1}^{n+m} x_i$$

$$A\tilde{x} + \bar{x} = b$$

$$x \geq 0.$$

<u>feasible basic solution</u> of <u>LP'</u>:

$$x_i = 0 \quad \text{for } 1 \leq i \leq n$$

$$x_{n+i} = b_i \quad \text{for } 1 \leq i \leq m.$$

basis: unit matrix $\underline{I}$.

The variables $x_{n+1}, x_{n+2}, \dots, x_{n+m}$ are called <u>artificial variables</u>.

<u>Question:</u>

What is the output of SIMPLEX with input LP' and the feasible basic solution above?

- If LP has no feasible solution then SIMPLEX terminates with a basic feasible solution which has positive values for some artificial variables; i.e. $\bar{x} \neq 0$.
- If LP has a feasible solution then SIMPLEX terminates with a basic feasible solution where all artificial variables are zero; i.e., $\bar{x} = 0$.
  - If no artificial variable is a basic variable then we obtain a basic feasible solution of LP by omitting the artificial variables
  - Otherwise, the solution is degenerate.

Goal:

Exchange of artificial basic variables by nonbasic variables which are not artificial

or

deletion redundant equations which contains artificial basic variables such that all remaining basic variables are not artificial. Then we obtain a basic feasible solution by omitting the artificial variables.

Suppose that the pth variable of B is artificial. Let e_p denote the pth column of the unit matrix. We distinguish two cases.

<u>Case 1:</u> There is a nonbasic variable x_q, $q \le n$ with $e_p^T B^{-1} a_q \neq 0$, i.e., $y_{pq} \neq 0$

Apply a pivot step to x_{B_p} and x_q; i.e., x_{B_p} is replaced by x_q.

$\Rightarrow$

We obtain a basic feasible solution of the same cost but with one artificial variable less.

<u>Case 2:</u> $e_p^T B^{-1} a_j = 0$; i.e. $y_{pj} = 0$ $\forall$ nonbasic variables x_j, $j \le n$.

$\Rightarrow$

By applying elementary row operations a zero-row has been constructed from the original matrix

$\Rightarrow$

$Ax = b$ is redundant such that the p-th column of B and the p-th row of A can be deleted. such that an artificial variable has been deleted from the basis.

Repeat this procedure until no basic variable is artificial.