

### 1.3. Strong connectivity

Let  $G = (V, E)$  be a directed graph. Two nodes  $v, u \in V$  are strongly connected if there is a path from  $v$  to  $u$  and also a path from  $u$  to  $v$  in  $G$ .

The relation "strongly connected" is an equivalence relation. Hence, we can partition  $V$  into the equivalence classes  $V_i$ ,  $1 \leq i \leq r$  with respect to the relation "strongly connected".

Let

$$E_i := \{(v, w) \in E \mid v, w \in V_i\}.$$

The graphs  $G_i := (V_i, E_i)$ ,  $1 \leq i \leq r$  are the strongly connected components of  $G = (V, E)$ .

$G$  is strongly connected if it has only one strongly connected component.

#### Properties:

- Each node in  $V$  is in exactly one strongly connected component.
- Edges with the property that its end nodes are in distinct connected components are in no strongly connected component. Such an edge is called 2c-edge.

The reduced graph  $G_{red} = (V_{red}, E_{red})$  is defined by

$V_{\text{red}} := \{ v_i \mid G_i = (V_i, E_i) \text{ is a strongly connected component} \}$

$E_{\text{red}} := \{ (v_i, v_j) \mid \exists u \in V_i, v \in V_j : (u, v) \in E \}$

Note that  $G_{\text{red}}$  is acyclic. Hence,  $G_{\text{red}}$  can be sorted topologically.

Goal:

The computation of the strongly connected components  $G_i = (V_i, E_i)$ ,  $1 \leq i \leq r$  of a given directed graph  $G = (V, E)$ .

Assume that  $G_{\text{red}}$  is topological sorted and that the strongly connected components are numbered with respect to the topological sorting of  $G_{\text{red}}$ .

$\Rightarrow$

If there is a 2c-edge from component  $V_i$  to component  $V_j$  then  $i < j$ .

Idea:

- (1)  $k := r$ ;
- (2) while  $k > 0$

do

- Compute an arbitrary node  $u_k \in V_k$ , the reference node for  $V_k$ ;
- $\text{DFS}(u_k)$ ;

od:  $k := k - 1$

The nodes visited during  $\text{DFS}(u_k)$  are exactly the nodes in  $V_k$ .

~)

backward construction of the strongly connected components of  $G$ .

Question:

How to obtain the reference nodes in the correct order?

We say that the nodes in  $G = (V, E)$  are well numbered if starting with number 1 a node  $v \in V$  obtains the next free number at the moment when the node  $v$  leaves the DFS-stack.

Lemma 1.10

Let  $G = (V, E)$  be a directed graph and let the nodes of  $G$  be well numbered by  $\text{num}: V \rightarrow \{1, 2, \dots, n\}$ . Then for all  $u, v \in V$  with  $\text{num}(u) > \text{num}(v)$  the following holds.

If there is in  $G$  a path  $P$  from  $v$  to  $u$  such that  $\text{num}(w) \leq \text{num}(u)$  for all  $w$  on  $P$  then both nodes  $u$  and  $v$  are in the same strongly connected component.

Proof:

We have to prove that there is a path from  $u$  to  $v$  in  $G$ .

Assume that there is no such a path in  $G$ .

$\Rightarrow$

$v$  is not a node in the subtree with root  $u$  of the DFS-tree.

$\text{num}(v) < \text{num}(u)$  and  $\nexists$  path from  $u$  to  $v$

At the moment when  $v$  enters the DFS-stack  
the node  $u$  cannot be already considered.

Let

$$P = v = w_1, w_2, \dots, w_k = u$$

Let  $i \geq 1$  maximal such that

- directly after the entering of  $v$  there holds:  $w_i$  is already considered.

Since  $v = w_i$  is already considered such  $i$  exists.  
Since  $u = w_k$  is not already considered, we have  $i < k$ . Two cases can arise:

Case 1:

At the moment when  $v$  enters the DFS-stack  
the node  $w_i$  is not in the stack.

6

Then  $w_i$  has left the DFS-stack before the entering of  $v$ . But before the performance of  $\text{Pop}(w_i)$  the edge  $(w_i, w_{i+1})$  is considered by DFS

$\Rightarrow$

$w_{i+1}$  is already considered

a contradiction

### Case 2:

$w_i$  is in the DFS-stack when  $v$  enters the stack.

Since there is a path from  $w_i$  to the node  $u$ ,  $\text{Push}(u)$  is performed by DFS before  $\text{Pop}(w_i)$  and hence, also  $\text{Pop}(u)$  is performed before  $\text{Pop}(w_i)$

$\Rightarrow$

$\text{num}(u) < \text{num}(w_i)$

a contradiction. ■

Let  $G' = (V, E')$  be the backward graph of  $G$ ; i.e.,

$$E' := \{(v, w) \mid (w, v) \in E\}.$$

### Observation

The strongly connected components of  $G$  and  $G'$  are the same.

Goal:

The development of a rule for the determination of the reference node for the computation of the strongly connected components of  $G'$ .

Lemma 1.11

Let  $G = (V, E)$  be a directed graph and let the nodes in  $V$  be well numbered by  $\text{num}: V \rightarrow \{1, \dots, n\}$ .  
Apply the following rule (R) for the determination of the reference nodes for the computation of the strongly connected components of  $G' = (V, E')$ .

(R) Choose the node  $u$  with  $\text{num}(u)$  is maximal within all nodes which are not assigned to a component.

Then  $u$  is in a strongly connected component such that all successor components with respect to the reduced graph  $G'_{\text{red}}$  are already constructed.

Proof:

(R)  $\Rightarrow$

For all nodes  $v$  visited during DFS on  $G'$  with start node  $u$  for the first time exists in  $G$  a path  $P$  from  $v$  to  $u$  with

$$\text{num}(w) \leq \text{num}(u) \quad \forall w \text{ on } P.$$

Lemma 1.10  $\Rightarrow$

If  $\text{num}(u) > \text{num}(v)$  then both nodes  $u$  and  $v$  are in the same connected component.

Otherwise, we obtain by rule (R) that the strongly connected component containing  $v$  is already constructed.



- DFS on  $G_i$  for the computation of  $\text{num}$ .
- DFS on  $G'_i$  for the computation of the strongly connected components.

### Theorem 1.10

Let  $G = (V, E)$  be a directed graph. Then we can compute the strongly connected components of  $G$  in time  $O(|V| + |E|)$ .