"On the Power of Two-Way Random Generators and the Impossibility of Deterministic Poly-Space Simulation"

by

Marek Karpinski *†

and

Rutger Verbeek *

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Abstract [BCP 83] proves that both probabilistic acceptors and transducers working in space $S(n) \geq \log n$ can be simulated by deterministic machines in $O(f(n)^2)$ space. The definition of probabilistic computations uses one-way read-only random tape. [BCP 83] asks: "Is it possible to extend our simulation results to the case of a two-way read-only oracle head?" In the same vein [FLS 83] suggests that it could be a difference between two-way and one-way random tape: " ... for space-bounded probabilistic computations where the space bound is much less than the length of y, it could matter." (y denoting the random tape inscription). In this paper we give a full characterization of two-way random space classes that answers both questions. We prove that there is no polynomial deterministic space simulation of two-way random space. In fact our result is stronger, saying that the probabilistic two-way random tape algorithms are precisely exponentially more powerful than the probabilistic one-way random tape algorithms.

* Department of Computer Science, University of Bonn, 53oo Bonn 1, W. Germany

† Department of Computer Science, University of Dortmund, 46oo Dortmund 5o, W. Germany

1. Two-way random tape

The model of probabilistic machine [Gi 77] may be viewed as a deterministic machine with a one-way read-only oracle head. The oracle tape records an unbounded sequence of outcomes of independent unbiased coin tosses. A random two-way oracle proposed in [BCP 83] is an unbounded random sequence recorded on a two-way read-only tape. Please, note that [BG 81] uses random oracle stored in a device resembling random access store rather than tape (i.e. questions must be written on a query tape within the space bound).

<u>Definition</u> Let $\psi \subseteq \Sigma^* \times \{0,1\}^\omega$ be a binary predicate, where $\psi(x,y)$ is computed by a deterministic machine M with two two-way read-only input tapes. If M stops on an initial segment of y, then $\psi(x,y)$ is defined. $x \in \Sigma^*$ is recognized by M if and only if $\Pr\{\psi(x,y) = \text{true}\} > \frac{1}{2}$. We call M a probabilistic machine (over the alphabet Σ) with two-way random tape. Let $L_M \subseteq \Sigma^*$ denote the set recognized by M.
If M is $S(|x|)$ space bounded, then L_M belongs to the two-way random-tape probabilistic space $S(n)$, $L_M \in Pr_2SPACE(S(n))$. If in addition M is $T(|x|)$ time bounded, then $L_M \in Pr_2TISP(T(n),S(n))$.
We say that L_M belongs to the two-way Las Vegas ([BGM 81]) Space $S(n)$, $L_M \in \Delta_2SPACE(S(n))$, if for all $x \in \Sigma^*$ either $\Pr\{\psi_M(x,y) = \text{true}\} = 1$ or $\Pr\{\psi_M(x,y) = \text{false}\} = 1$.

<u>Remarks</u> 1. If M reads the second input tape one-way, then this model is equivalent to the classical model of probabilistic machines [Gi 77]. We denote these classes by $Pr_1SPACE(S(n))$, $Pr_1TISP(T(n),S(n))$, etc.

2. If M is time bounded, then y needs not to be infinite and $\Pr\{\psi(x,y) = \text{true}\}$ can be replaced by $|\{y : |y| = T(|x|) \text{ and } \psi(x,y) = \text{true}\}| / 2^{T(|x|)}$ [FLS 83].

3. The random tape y is unbounded only to the right, but all simulation results in this paper can be easily extended to the case of random tapes, that are unbounded in both directions. We do not know, however, whether the two models are equivalent for very small space bounds.

4. Denote by $DSPACE_2^A(f(n))$ the class of sets recognized by deterministic oracle-machines with two-way oracle tape A. Then, with probability 1 (i.e. for almost all oracles), $DSPACE_2^A(f(n)) \supsetneq \Delta_2SPACE(f(n))$ (the inequivalence results from the fact that the set of positions where A contains 1 is clearly in $DSPACE_2^A(O(1))$ but, with probability 1, not in $\Delta_2SPACE(f(n))$).

2. Main results

It is not obvious that Pr_2SPACE and Δ_2SPACE define Blum complexity measures. On the other hand it was not known whether Pr_2SPACE is more powerful than Pr_1SPACE. Our characterization settles both questions:

Theorem 1 If $S(n) \geq \log n$, then
$\Delta_2\mathrm{SPACE}(S(n)) = \mathrm{Pr}_2\mathrm{SPACE}(S(n)) = \bigcup_c \mathrm{Pr}_1\mathrm{SPACE}(c^{S(n)}) = \bigcup_c \mathrm{DSPACE}(c^{S(n)})$.
In particular we have $\Delta_2\mathrm{SPACE}(\log n) = \mathrm{PSPACE}$.

This theorem gives also a negative answer to question of [BCP 83]:
Corollary (Impossibility of deterministic poly-space simulation)
For all functions $S(n) \geq \log n$ and all $k \in \mathbb{N}$
$\mathrm{Pr}_2\mathrm{SPACE}(S(n)) \nsubseteq \mathrm{DSPACE}(S(n)^k)$.

Theorem 1 is related to the recent result of [DS 84] that 'consistent' NSPACE (CSPACE) is exponentially more powerful than DSPACE. The similarity becomes clear, if the reset mechanism in the original definition of CSPACE is replaced by a two-way tape, on which the initial nondeterministic choices are recorded. The proof of our lemma 4 can be applied to this case (replace Pr_2 by consistent N and Pr_1 by N). Our method of the proof of lemma 1 and 4 yields also a characterization of CSPACE(O(1)): $\mathrm{NSPACE}(n) \subseteq \mathrm{CSPACE}(O(1)) \subseteq \mathrm{NSPACE}(n \log n)$ for both definitions of CSPACE. Our results for Pr_2-classes can be proved also for reset random tapes instead of two-way random tapes, but the results for Δ_2 cannot (to our knowledge).

For the simultaneous time and space bounded classes we can reduce the space bound exponentially using a two-way random tape:

Theorem 2 If $f(n) \geq n$ then
$\mathrm{Pr}_2\mathrm{TISP}(\mathrm{poly}(f(n)), \log f(n)) = \mathrm{Pr}_1\mathrm{TIME}(\mathrm{poly}(f(n)))$, in particular
$\mathrm{Pr}_2\mathrm{TISP}(\mathrm{poly}, \log) = \mathrm{Pr}_1\mathrm{TIME}(\mathrm{poly}) = \mathrm{PP}$.

3. Probabilistic space

In this section we will prove theorem 1 and give characterizations of the power of finite automata with two-way random tape, which are summarized in a table at the end of the section.

Lemma 1 $\Delta_2 SPACE(f(n)) \supseteq \bigcup_c DSPACE(n \cdot c^{f(n)})$ for all functions $f(n)$.

Proof Suppose M is a deterministic $n \cdot c^{f(n)}$ space bounded single-tape DTM that halts on every input. Let $comp_M(x) = ¢\, q_o\, x\, \$\, c_1\, \$\, c_2\, \$\, \ldots\, ¢ \in \bar{\Sigma}^*$ denote the computation of M on input $x \in \Sigma^*$. Let $h : \bar{\Sigma} \to \{0,1\}^k$ be a binary encoding of $\bar{\Sigma}$ (for appropriate k). Then the relation $\rho_M := \{(x,y) \mid x \in \Sigma^*,\ y \in (\{0,1\}^k)^*\, h(comp_M(x))\{0,1\}^*\}$ is accepted by some $f(|x|)$ space bounded machine M'. Let M' search on y for $h(¢)$ and then verify $h(q_o x)$ using the input x. If the first instantaneous description (ID) is verified, M' compares it with c_1, then c_1 with c_2, etc. The comparisons are made using the first input tape (containing x) and the work tape to simulate a counter of maximal size $k \cdot n \cdot c^{f(n)}$. If a comparison fails, M' will look (to the right) for the next substring $h(¢)$ and restart the process.
Let $\psi \in \Sigma^* \times \{0,1\}^\omega$ be the following predicate:

$$\psi(x,y') = \begin{cases} \text{true if an initial segment } y \text{ of } y' \text{ exists with } \rho_M(x,y) \text{ and } x \in L_M \\ \text{false if } \rho_M(x,y) \text{ for an initial segment } y \text{ and } x \notin L_M \\ \text{undefined otherwise.} \end{cases}$$

ψ is computed by a DTM M'', which simulates M' until y is found such that $\rho_M(x,y)$. Then M'' checks whether the final ID in $comp_M(x)$ is accepting or rejecting. M'' is $f(n)$ space bounded and yields the correct answer, if it stops. Since $Pr\{\rho_M(x,y)$ for some initial segment$\} = 1$, M'' will stop with probability 1. The expected running time is $c^{|comp_M(x)|}$.

□

Remark Lemma 1 is valid also for transducers: M'' will produce the output after verifying the complete computation.

Lemma 2 $\bigcup_c Pr_2 TISP(2^{n \cdot c^{f(n)}}, f(n)) \supseteq \bigcup_c NSPACE(n \cdot c^{f(n)})$.

Proof The only difference to the proof of lemma 1 is the definition of ρ_M and $\psi_{M''}$. Define $\rho_M = \{(x,y) \mid x \in \Sigma^*,\ y \in \{0,1\}\, h(comp_M(x))\{0,1\}^*\}$, where M is a strictly $n \cdot c^{f(n)}$ space bounded NTM which stops on every input x. Since M is nondeterministic, $comp_M(x)$ is not unique. Define $\psi_{M''}(x,y') \iff ((x,y) \in \rho_M$ for some initial substring y of y' and $comp_M(x)$ is accepting) or y' starts with 1. Obviously M'' stops after $k \cdot |comp_M(x)| \le \bar{c}^{n \cdot c^{f(n)}}$ steps.
$Pr\{\psi_{M''}(x,y) = \text{true}\} = Pr\{y$ starts with $1\} + Pr\{y$ starts with 0 and an initial substring of y is the encoding of an accepting computation on $x\} > \frac{1}{2} \iff x \in L_M$.

□

For the next lemmas we use a variation of probabilistic machines with two-way random tape. These machines can test whether the head on the random tape scans the rightmost square of the tape it had visited until this step. The machines behave like probabilistic auxiliary nonerasing stack automata where probabilistic choices are allowed during push-steps only. We denote the corresponding complexity classes by $\overline{\Delta}_2$SPACE, $\overline{Pr}_2$SPACE, and $\overline{Pr}_2$TISP.

<u>Lemma 3</u> (1) $\overline{\Delta}_2\mathrm{SPACE}(O(1)) \supseteq \mathrm{DSPACE}(n \log n)$,

(2) $\bigcup_c \overline{Pr}_2\mathrm{TISP}(c^{n \log n}, O(1)) \supseteq \mathrm{NSPACE}(n \log n)$.

<u>Proof</u> (1) Suppose $L \in \mathrm{DSPACE}(n \log n)$. Then L is recognized by some deterministic nonerasing stack automaton M halting on every input (see [HU 79]). We encode the working alphabet $\overline{\Sigma}$ of M by $h : \overline{\Sigma} \to \{0,1\}^k$. Let $\mathrm{comp}_M(x)$ denote the final stack inscription which M produces on input x with bottom marker ¢. Let $\psi_{M'}(x,y) \iff y \in \{0,1\}^k \mathrm{comp}_M(x)\{0,1\}^\omega$ and $x \in L_M$. $\psi_{M'}$ can be computed by a deterministic finite automaton which can test, whether it scans the rightmost square of y visited in a preceding step. As in lemma 1, $\Pr\{M' \text{ stops on } x\} = 1$ and M' will always make the correct decision.

(2) The same construction as in the deterministic case [HU 79] shows that all sets in $\mathrm{NSPACE}(n \log n)$ can be recognized by halting nondeterministic nonerasing stack automata with nondeterminism restricted to push-moves and stack length $c^{n \log n}$. Using the techniques of lemma 2 and 3(1) these stack automata are simulated in $\overline{Pr}_2\mathrm{TISP}(\bar{c}^{n \log n}, O(1))$.

□

Lemmas 1 - 3 yield surprising high lower bounds for the Pr_2 and Δ_2 space classes. The next lemma shows that these lower bounds are (almost) optimal.

<u>Lemma 4</u> (1) $\overline{Pr}_2\mathrm{SPACE}(f(n)) \subseteq \bigcup_c Pr_1\mathrm{SPACE}(n \log n \cdot c^{f(n)})$

(2) $\overline{\Delta}_2\mathrm{SPACE}(f(n)) \subseteq \bigcup_c \Delta_1\mathrm{SPACE}(n \log n \cdot c^{f(n)})$.

<u>Proof</u> Suppose M is a $\overline{Pr}_2\mathrm{SPACE}(f(n))$-machine (not necessarily halting). The simulation is almost the same as the simulation of (deterministic) nonerasing auxiliary stack automata by space bounded Turing machines. The number of configurations of M is bounded by $c_1 \cdot n \cdot c_2^{f(n)} =: C$. We associate with every position i on the random tape a table T_i with C entries $T_i(c)$. Each $T_i(c)$ gives full information about the behavior of M for the case that M starts in configuration c on random tape position i and moves left:

$T_i(c) = c'$ $\iff$ if M starts at i in configuration c and moves left, then it will come back to position i in configuration c'

$T_i(c) = \text{accept}$ $\iff$ if M starts at i in configuration c and moves left, then it will stop and accept before coming back

$T_i(c) = \text{reject}$ $\iff$ if M start at i in c to the left, then it will neither come back nor accept (i.e. either reject or cycle).

A table T_i can be stored in space $C \cdot \log C \leq n \cdot \log n \cdot c^{f(n)}$ for appropriate c. The table T_o is trivial and T_{i+1} can be computed from T_i (and M's table) using the (i+1)st random bit. Thus all the left-moves of M on the random tape can be simulated by looking at the actual table. If the simulated machine is a $\overline{\Delta}_2$-machine, then the simulating machine is Δ_1, since the probabilities are not affected by the simulation.

□

<u>Corollary</u> Δ_2, $\overline{\Delta}_2$, Pr_2, $\overline{Pr}_2$ are Blum complexity measures. □

Combining lemmas 1 and 4 with the deterministic $f(n)^2$-space simulation of [BCP 83] we get the following characterization:

<u>Theorem 1</u> If $f(n) \geq \log n$, then

$$\Delta_2\text{SPACE}(f(n)) = Pr_2\text{SPACE}(f(n)) = \overline{Pr}_2\text{SPACE}(f(n)) = \bigcup_c Pr_1\text{SPACE}(c^{f(n)}) = \bigcup_c \text{DSPACE}(c^{f(n)}).$$

<u>Proof</u> If $f(n) \geq \log n$, then $n \cdot \log n \cdot c^{f(n)} \leq O(c^{f(n)})$. Thus

$$\begin{aligned}
&\bigcup_c \text{DSPACE}(c^{f(n)}) \subseteq \Delta_2\text{SPACE}(f(n)) && \text{(lemma 1)}\\
&\subseteq Pr_2\text{SPACE}(f(n)) \subseteq \overline{Pr}_2\text{SPACE}(f(n)) && \text{(obvious)}\\
&\subseteq \bigcup_c Pr_1\text{SPACE}(c^{f(n)}) && \text{(lemma 4)}\\
&\subseteq \bigcup_c \text{DSPACE}(c^{f(n)}) && \text{([Ju 81], [BCP 83])}.
\end{aligned}$$

□

<u>Remark</u> Theorem 2 suggests that two-way random tape machines are much more powerful than even alternating space bounded machines [CKS 81]:

$$\mathrm{ASPACE}(f(n)) = \bigcup_c \mathrm{DTIME}(c^{f(n)}) \overset{(\neq?)}{\subseteq} \bigcup_c \mathrm{DSPACE}(c^{f(n)}) = \bigcup_c \mathrm{ATIME}(c^{f(n)}) =$$

Δ_2 SPACE(f(n)) for $f(n) \geq \log n$.

If $f(n)$ is $o(\log n)$ then they are provably more powerful (lemma 1).

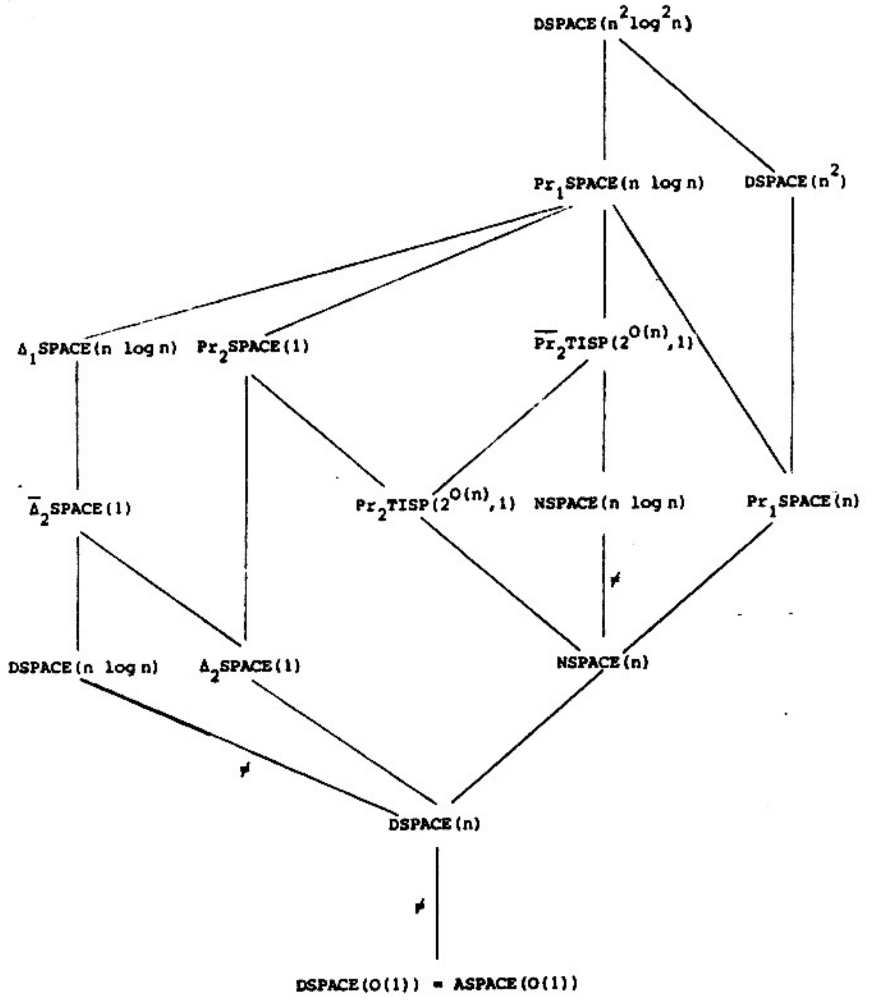
In the case of small space bounds the situation is more complex. The inclusions

$$\mathrm{NSPACE}(n \log n) \subseteq \overline{\mathrm{Pr}}_2\mathrm{SPACE}(O(1)) \subseteq \mathrm{Pr}_1\mathrm{SPACE}(n \log n) \quad \text{(lemmas 3,4)}$$

give evidence, that the bounds cannot be improved.

We summarize the situation for constant space in the following diagram

($\begin{matrix} B \\ | \\ A \end{matrix}$ means $A \subseteq B$):



4. Probabilistic TISP

In this chapter we show that even for time bounded computation the use of a two-way random tape can drastically reduce the space bounds.

Let PL_2 denote the sets recognized simultaneously in polynomial time and log space by probabilistic machines with two-way random tape, i.e. $PL_2 = Pr_2TISP(poly, log) \subseteq Pr_2TIME(poly)$. PP [Gi 77] stands for the probabilistic polynomial time class, $PP = Pr_1TIME(poly)$.

Lemma 5 If $T(n) \geq n$ and $T(n)$ is computed in $DTISP(T(n)^3, \log T(n))$, then $Pr_2TISP(T(n)^3, \log T(n)) \supseteq Pr_1TIME(T(n))$.

Proof Denote by M a PTM with one-way random tape recognizing X in $T(n)$ time. The proof follows the proof of lemma 2 with some minor changes. Let $comp_M(x)$ denote the set of computations of M on x encoded by $¢\, c_o\, \alpha_o\, \$\, c_1\, \alpha_1\, \$\, c_2\, \alpha_2\, \$ \ldots \$\, c_{T(n)}\, ¢$ where the c_i's are encodings of IDs padded with blanks to exactly the same length $(k \cdot T(n))$ and the α_i's are the random choices in the computation (i.e. the first $T(n)$ bits of M's random tape) such that $c_i \vdash_M c_{i+1}$ reading α_i on the random tape. A stopping ID c_k, $k < T(n)$, is identically repeated up to the step $T(n)$ (with arbitrary choices of α_i's). Let t denote the exact length of an appropriate binary encoding $h(comp_M(x))$ as defined in the proof of lemma 1. Denote by y^t the initial segment of length t of any $y \in \{0,1\}^\omega$.

Define the predicate ψ by

$$\psi(x,y) \iff (y^t \text{ describes an accepting computation of } M \text{ on } x) \text{ or }$$
$$(y^t \notin h(comp_M(x)) \text{ and the } (t+1)\text{st bit of } y \text{ is } 1).$$

Then ψ is computed by some DTM M' working in time $T(|x|)^3$ and space $\log(T(|x|)^3) = O(\log T(|x|))$. M' recognizes X because

$$\begin{aligned}
\Pr\{\psi_{M'}(x,y) = \text{true}\} &= \Pr\{h^{-1}(y^t) \in \text{comp}_M(x) \text{ and } h^{-1}(y^t) \text{ accepting}\} \\
&+ \tfrac{1}{2}\Pr\{h^{-1}(y^t) \notin \text{comp}_M(x)\} \\
&= \Pr\{h^{-1}(y^t) \in \text{comp}_M(x)\} \cdot \Pr\{M \text{ accepts } x\} \\
&+ \tfrac{1}{2}(1-\Pr\{h^{-1}(y^t) \in \text{comp}_M(x)\}) \\
&= \tfrac{1}{2} + \Pr\{h^{-1}(y^t) \in \text{comp}_M(x)\} \cdot (\Pr\{M \text{ accepts } x\} - \tfrac{1}{2}) \\
&< \tfrac{1}{2} \iff x \in X.
\end{aligned}$$

(The careful encoding of $\text{comp}_M(x)$ guarantees the equality
$\Pr\{h^{-1}(y^t) \in \text{comp}_M(x)$ and $h^{-1}(y^t)$ accepting$\}$
$= \Pr\{h^{-1}(y^t) \in \text{comp}_M(x)\} \cdot \{M$ accepts $x\}$.)

□

Theorem 2 For all functions $f(n)$ which can be computed in
$\text{DTISP}(\text{poly}(f(n)), \log f(n))$,
$\Pr_2\text{TISP}(\text{poly}(f(n)), \log f(n)) = \Pr_1\text{TIME}(\text{poly}(f(n)))$ $(= \Pr_2\text{TIME}(\text{poly}(f(n))))$.
In particular $PL_2 = PP$.

Proof From Lemma 5 we have
$\Pr_2\text{TISP}(\text{poly}(f(n)), \log f(n)) \supseteq \Pr_1\text{TIME}(\text{poly}(f(n)))$.
Since the space is unbounded, we can store the random sequence. Thus
$\Pr_1\text{TIME}(\text{poly}(f(n))) \supseteq P_2\text{TIME}(\text{poly}(f(n)))$.

We denote by $\Pr_2SC^k$ the two-way random tape analogon of deterministic SC^k classes ([Co 79], [Ru 81]), meaning simultaneous poly-time and $\log^k n$ space, i.e. $\Pr_2SC^k = \Pr_2\text{TISP}(\text{poly}, \log^k)$, $\Pr_2SC = \bigcup_k \Pr_2SC^k$.

Corollary $\Pr_2SC = \Pr_2SC^1$.

Proof by theorem 2.

□

Theorem 3 $\text{NTISP}(\text{poly}, n) \subseteq \Pr_2\text{TISP}(\text{poly}, O(1)) \subseteq \Pr_1\text{TISP}(\text{poly}, n \log n)$.

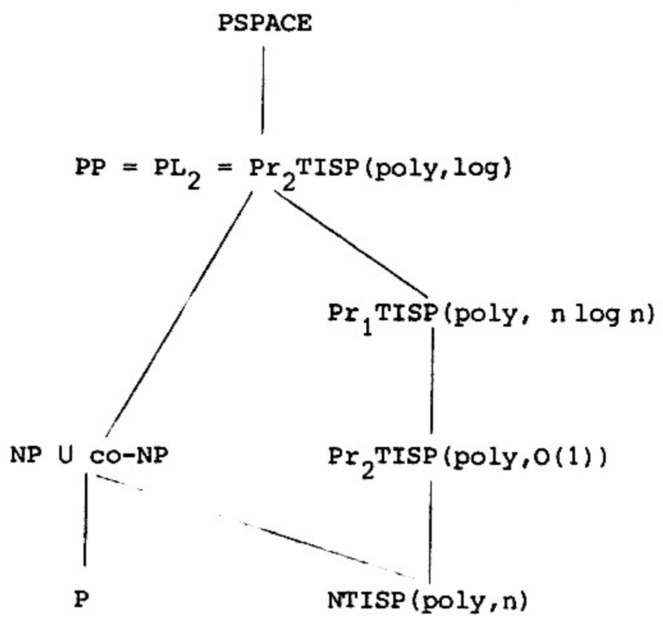
Proof Apply the simulation of lemma 2 to a nondeterministic machine whose all computations are polynomial time bounded. The second inclusion is by the construction of lemma 4.

□

Remark Using the notation of [DS 84] and a reset tape y instead of a two-way tape we get

$NTISP(poly,n) \subseteq CTISP(poly,O(1)) \subseteq NTISP(poly, n\log n)$.

The theorems above yield the following diagram:



5. Conclusions

1. [KV 84] proves the impossibility of subexponential deterministic simulation of a two-way random tape solving the open problem of [BCP 83].

   At this point we did not know whether or not Pr_2SPACE is a Blum complexity measure. In particular we asked the question on whether $Pr_2SPACE(O(1)) \subseteq DSPACE(h(n))$ for some function h.

   This paper gives a tight bound almost meeting our lower bound. The function h we have got is $n^2\log^2 n$. An interesting question would be whether two-way random tape Monte Carlo TISP(poly,O(1)) includes nonregular languages. We know by [GW 84] that this is impossible for the one-way case.

2. The results on two-way random tape entail nonexistence of the probabilistic analogon to polynomial-expanding PNGs ([BM 82], [FLS 83]). For, the existence of polynomial-expanding PNGs that cannot be distinguished from an ideal (physical) random number generator-would entail PSPACE = DSPACE(lin) by the proof of lemma 1, the impossiblility.

   The technical realization of a two-way random tape lies near at hand, in what one records Bernoulli trials in secondary machine storage first and afterwards runs the algorithm.

3. Theorem 1 proofs an exponential (in the space bound) lower bound for the depth of uniform circuits [BCP 83] simulating the two-way random tape. This contrasts with Borodin-Cook-Pippenger NC^2-circuits developed for the simulation of one-way random tape.

4. One can show that UNIQUE SAT [BG 82], [PY 82] is in $Pr_2TISP(poly,O(1))$. Therefore UNIQUE SAT is in $PL_2 = PP$. Proving that SAT - UNIQUE SAT (i.e. given two formulas F and G, decide whether F is satisfiable and G is uniquely satisfiable) is in PL_2 would yield $D^P \subseteq PL_2 = PP$ [BG 82], [PY 82]. If UNIQUE SAT is complete in D^P, then SAT - UNIQUESAT ε PL_2.

5. The simulations of lemmas 3 and 4 are tight unless $NSPACE(n \log n) \subseteq Pr_1SPACE(o(n \log n))$ which would improve Savitch's [Sa 7o] well known simulation by the Borodin-Cook-Pippenger [BCP 83] deterministic squared space simulation.

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