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Multivariate Polynomials, Standard Tableaux, and Representations of Symmetric Groups

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This paper is concerned with structural and algorithmic aspects of certain R-bases in polynomial rings $R[X_{ij}]$ over a commutative ring R with 1. These bases are related to standard tableaux. We shall examine the main tools in full detail: (symmetrized) bideterminants, Capelli operators, hyperdominance, and generalized Laplace's expansions. These tools are then applied to the representation theory of symmetric groups. In particular, we present an algorithm which efficiently computes for every skew module of a symmetric group an R-basis which is adapted to a Specht series. This result is a constructive, characteristic-free analogue of the celebrated Littlewood-Richardson rule. This paper will serve as the basis for a possible generalization of that rule to more general shapes.

1 Introduction

This paper is concerned with structural and algorithmic aspects of bases in polynomial rings, which are related to standard tableaux. It is closely related to the works of Doubilet *et al.* (1974), Désarménien *et al.* (1978), De Concini *et al.* (1980) and Clausen (1979,1980). For historical remarks the interested reader is referred to these papers.

To be more specific let R be a commutative ring with unit element $1_R \neq 0$. The polynomial ring $R[X_{ij}]$ is a free R-module. Its most commonly used R-basis consists of (normalized) monomials in the indeterminates X_{ij} $(i, j \in$

$\mathbb{N} := \{1,2,\ldots\}$). In this paper we study other R-bases, which seem to be appropriate for tackling a number of structural and algorithmic problems, e.g. in invariant theory, representation theory, algebraic geometry, commutative algebra, physics and chemistry; see the references for more information. The bases in question are closely related to certain pairs of standard tableaux. In fact, the possible linear transformations from the basis of monomials to the latter bases can be viewed as linear analogues of the celebrated Robinson–Schensted–Schützenberger–Knuth correspondence in combinatorics, cf. Schensted (1961), Schützenberger (1963), and Knuth (1970).

Surprisingly, the whole subject is essentially based on two tools: generalized Laplace's expansions and Capelli operators. In this paper we give a thorough treatment of these tools with emphasis on the group theoretical and combinatorial background. The Laplace Duality Theorem, cf. section 4, shows that rather different-looking polynomial expressions in minors of the matrix (X_{ij}) can define exactly the same polynomial in $R[X]$. This result, which extends and simplifies the generalized Laplace's expansions in Désarménien *et al.* (1974), Clausen (1980a) and (1980b), is the basis of several straightening formulae and determinantal identities, cf. Doubilet *et al.* (1974), Désarménien (1980), de Concini *et al.* (1980), and Abhyankar (1988).

Section 3 presents a new approach to Capelli operators, which avoids the introduction of a set of new ("coloured") indeterminates, cf. Désarménien *et al.* (1978), Clausen (1980b). A closer look at the Capelli operators leads to a partial ordering of standard tableaux, which we call hyperdominance, since this ordering is contained in the well-known dominance partial ordering of standard tableaux. Using hyperdominance, we get stronger results and simplified proofs. Compared to the dominance partial ordering these facts indicate that hyperdominance is a more natural "data structure" in this context. Sections 2 and 5 discuss bideterminants and symmetrized bideterminants, and section 6 shows that they are adjoint to each other.

Finally we apply these tools to representation theory. Besides other possible applications to the modular representation theory of classical groups, see e.g. Clausen (1979), (1980b), Green (1980), Golembiowski (1987), and Pittaluga & Strickland (1988), we concentrate on the decomposition of certain cyclic RS_n-modules. More precisely, let G be a finite permutation group. Then G acts on $R[X_{ij}]$ as a group of R-algebra automorphisms via $\pi X_{i,j} := X_{\pi(i),j}$. For a number of classical groups and suitable rings R parts of the bases mentioned above reflect on one side the structure of simple RG-modules. On the other side, the right hand indices, j, in the X_{ij} help via the Laplace Duality Theorem to systematically generate combinatorial structures, which count the multiplicities of the simple constituents. This program works for various series of classical groups and will be illustrated by one example: the series of symmetric groups. Section 8 describes the simple modules for symmetric groups as Specht modules. Combining all the tools mentioned above, we can construct for every skew RS_n-module an R-basis that is adapted to a Specht series of this module. Section 9 presents the relevant proofs and section 10 describes an algorithm (joint work with F. Stötzer) which efficiently generates such a basis for every skew module. This makes results of James (1977), James & Peel (1979), and Zelevinsky (1981a) more precise. Our results will serve as the basis for a possible generalization of the Littlewood-Richardson rule to more general shapes, see Clausen & Grabmeier (1990), including a revision of the notion of standardness.

The structure of skew modules is closely related to the Clebsch-Gordan coefficients in quantum physics, see e.g. Dirl & Kasperkovitz (1977), to the multiplication of Schur functions, see e.g. Macdonald (1979) and Stanley (1971), as well as to the Schubert calculus, see e.g. Stanley (1977).

To make this paper essentially self-contained we give a brief introduction to the ordinary representation theory of finite groups in section 7. We start discussing several fundamental concepts, which will frequently be used in later sections. The polynomial ring $\mathbb{Z}[X] := \mathbb{Z}[X_{ij} : i,j \in \mathbb{N}]$ or suitable scalar extensions will serve as the universe where all our considerations will take place.

2 Bideterminants

In this section we describe a remarkable $\mathbb{Z}$-basis of $\mathbb{Z}[X]$ consisting of standard bideterminants. Bideterminants are power products in minors of the matrix (X_{ij}). Such a power product of minors is denoted by writing the factors along successive columns:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 8 & 1 & 2 & 2 \\ 1 & 2 & & 8 & 1 & \\ & 3 & & & 4 & \end{array}\right) = \left(\begin{array}{c|c} 2 & 1 \\ 1 & 8 \end{array}\right) \cdot \left(\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{array}\right) \cdot \left(\begin{array}{c|c} 8 & 2 \end{array}\right)$$

$$= \det\begin{pmatrix} X_{21} & X_{28} \\ X_{11} & X_{18} \end{pmatrix} \cdot \det\begin{pmatrix} X_{12} & X_{11} & X_{14} \\ X_{22} & X_{21} & X_{24} \\ X_{32} & X_{31} & X_{34} \end{pmatrix} \cdot X_{82}.$$

We prepare a more formal definition of bideterminants: Let A be a finite subset of $\mathbb{N} \times \mathbb{N}$. A mapping $T: A \to \mathbb{N}$ is called a *tableau* of *shape* A (or, an A-*tableau*) of *content* $(|T^{-1}\{1\}|, |T^{-1}\{2\}|, \ldots)$. As usual, we think of T as an A-shaped matrix (t_{ij}), where for $(i,j) \in A$, $t_{ij} := T((i,j))$ is the *entry* in *row* i and *column* j. $A \cap (\{i\} \times \mathbb{N})$ and $A \cap (\mathbb{N} \times \{j\})$ are the i-th *row* and j-th *column* of A, respectively. $\mathrm{Sym}(A)$ denotes the symmetric group on A, $\mathcal{H}(A) \simeq \prod_i \mathrm{Sym}(i\text{−th row of } A)$ is the subgroup of all *horizontal* permutations of A, its counterpart is $\mathcal{V}(A) \simeq \prod_j \mathrm{Sym}(j\text{−th column of } A)$, which is the group of all *vertical* permutations on A. An *A-bitableau* of *content* (α, β) is a pair, (S, T), of A-tableaux such that $\alpha = \mathrm{content}(S)$ and $\beta = \mathrm{content}(T)$. We call $\{S|T\} := \prod_{a \in A} X_{S(a),T(a)}$ the *natural monomial* and $(S|T) := \sum_{\sigma \in \mathcal{V}(A)} \mathrm{sgn}(\sigma)\, \{S \circ \sigma|T\}$ the *bideterminant* corresponding to the A-bitableau (S, T). We summarize some simple facts.

Lemma 2.1 *Let (S, T) and (U, V) be bitableaux of shapes A and B, respectively. Then*

(a) $\{S|T\} = \{U|V\}$ *if and only if there is a bijection* $\varphi: A \to B$ *such that* $(U \circ \varphi, V \circ \varphi) = (S, T)$.

(b) $\{S \circ \sigma|T\} = \{S|T \circ \sigma^{-1}\}$, *for all* $\sigma \in \mathrm{Sym}(A)$.

(c) $\sum_{\sigma \in \mathcal{V}(A)} \mathrm{sgn}(\sigma)\{S \circ \sigma|T\} = (S|T) = \sum_{\tau \in \mathcal{V}(A)} \mathrm{sgn}(\tau)\{S|T \circ \tau\}$.

(d) $(S|T) = \prod_j (S^j|T^j)$, *where the factor* $(S^j|T^j)$ *denotes the bideterminant (=minor) corresponding to the j-th columns of S and T.*

(e) $(S \circ \sigma|T \circ \tau) = \mathrm{sgn}(\sigma\tau)\,(S|T)$, *for all* $\sigma, \tau \in \mathcal{V}(A)$.

(f) $(S|T) \neq 0$ *iff all S^j and all T^j are injective. (In that case S and T are called* column-injective.*)* □

An A-tableau S is called *standard* iff A is a *diagram* (i.e. if $(i, j) \in A$ then $(i', j') \in A$ for all $1 \leq i' \leq i$ and $1 \leq j' \leq j$), and the entries in S are weakly increasing from left to right in each row, and strictly increasing from top to bottom in each column. A bitableau (S, T) is *standard*, iff both S and T are standard. By definition, the empty bitableau is standard, and $\{\emptyset|\emptyset\} = (\emptyset|\emptyset) := 1$.

Example.

$$\begin{pmatrix} 1 & 1 & 2 & 4 & & 2 & 3 & 3 & 5 \\ 2 & 3 & 5 & & , & 4 & 5 & 5 & \end{pmatrix}$$

is a standard bitableau of content $(\alpha, \beta) = ((2, 2, 1, 1, 1), (0, 1, 2, 1, 3))$. □

For a proof of the following crucial result the reader is referred to Désarménien *et al.* (1978). This result is due to Mead (1972), although parts or variants of it can be traced back to the works of Young, Turnball, Hodge, Igusa, among others.

Theorem 2.2 *The bideterminants corresponding to all standard bitableaux form a* $\mathbb{Z}$*-basis of* $\mathbb{Z}[X]$. $\square$

Later on we will need local versions of this theorem adapted to the following direct decomposition of $\mathbb{Z}[X]$ into $\mathbb{Z}$-submodules $\mathbb{Z}_{\alpha\beta}$ of finite rank:

$$\mathbb{Z}[X] = \bigoplus_{d \geq 0} \mathbb{Z}[X]_d = \bigoplus_{d \geq 0} \bigoplus_{\alpha,\beta} \mathbb{Z}_{\alpha\beta}.$$

Here, $\mathbb{Z}[X]_d$ is the space of all d-homogeneous polynomials, and for non-negative integral sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ satisfying $|\alpha| := \sum \alpha_i = d = |\beta|$, we define $\mathbb{Z}_{\alpha\beta}$ to be the $\mathbb{Z}$-span of all monomials $X_{i_1 j_1} \cdot \ldots \cdot X_{i_d j_d}$ of *content* (α, β) , i.e. $((i_1, \ldots, i_d), (j_1, \ldots, j_d))$ is a bitableau of content (α, β). We describe a $\mathbb{Z}$-basis of $\mathbb{Z}_{\alpha\beta}$: If $M_{\alpha\beta}$ denotes the (finite) set of all non-negative integral matrices with row (resp. column) sums $\alpha_1, \alpha_2, \ldots$ (resp. $\beta_1, \beta_2 \ldots$), then the monomials $X^M := \prod X_{ij}^{m_{ij}}$, corresponding to the elements $M = (m_{ij})$ in $M_{\alpha\beta}$, form a $\mathbb{Z}$-basis of $\mathbb{Z}_{\alpha\beta}$. If $\mathrm{SBT}(\alpha, \beta)$ denotes the set of all standard bitableaux of content (α, β), and $\mathrm{SBD}(\alpha, \beta)$ the corresponding set of bideterminants, then the local version of the above theorem gives a second $\mathbb{Z}$-basis of $\mathbb{Z}_{\alpha\beta}$.

Theorem 2.2 (local version) *The bideterminants corresponding to all standard bitableaux of content* (α, β) *form a* $\mathbb{Z}$*-basis of* $\mathbb{Z}_{\alpha\beta}$*, for short:* $\mathbb{Z}_{\alpha\beta} = \langle\langle \mathrm{SBD}(\alpha, \beta) \rangle\rangle_{\mathbb{Z}}$.

Observe that all results of this section remain valid if $\mathbb{Z}$ is replaced by any commutative ring R. In particular, $R_{\alpha\beta} = \langle\langle \mathrm{SBD}(\alpha, \beta) \rangle\rangle_R$.

Example. $\alpha = (2,3),\ \beta = (1,1,3)$.

$$\begin{aligned}
\mathbb{Z}_{\alpha\beta} &= \langle\langle X^{\binom{002}{111}}, X^{\binom{011}{102}}, X^{\binom{101}{012}}, X^{\binom{110}{003}} \rangle\rangle_{\mathbb{Z}} \\
&= \langle\langle (11222|12333), \left(\begin{array}{l|l} 1122 & 1333 \\ 2 & 2 \end{array} \right), \left(\begin{array}{l|l} 1122 & 1233 \\ 2 & 3 \end{array} \right), \\
&\qquad \left(\begin{array}{l|l} 112 & 123 \\ 22 & 33 \end{array} \right) \rangle\rangle_{\mathbb{Z}}.
\end{aligned}$$

$\square$

According to the last theorem every bideterminant $(U|V) \in \mathbb{Z}_{\alpha\beta}$ can uniquely be written as a $\mathbb{Z}$-linear combination of standard bideterminants:

$$(U|V) = \sum_{(S,T) \in \mathrm{SBT}(\alpha,\beta)} a_{ST,UV}\ (S|T).$$

The coefficients $a_{ST,UV} \in \mathbb{Z}$ have been called *straightening coefficients.* A problem which will frequently occur is to have *a priori* information about the (non-)vanishing of these straightening coefficients. The next section is concerned with this problem.

3 Capelli Operators

Let us first survey the content of this section: With every A-bitableau (S,T) we associate the so-called *Capelli operator* $C_{ST} \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[X]_d)$, $d = |A|$. A closer look at the Capelli operators leads to a relation $\mathrel{\hookleftarrow\!\Uparrow}$ on the set of bitableaux which is reflexive and antisymmetric when restricted to standard bitableaux. Hence its transitive closure $\mathrel{\hookleftarrow\!\Uparrow}^*$ is a partial ordering on the set of all standard bitableaux, the so-called *hyperbidominance.* Capelli operators, bideterminants and hyperbidominance are related by various fundamental properties; for the moment let us mention the following:

$$C_{ST}(S|T) \neq 0 \tag{1}$$

$$C_{ST}(U|V) \neq 0 \Rightarrow (S,T) \mathrel{\hookleftarrow\!\Uparrow} (U,V), \tag{2}$$

for all $(S,T),(U,V) \in \mathrm{SBT}(\alpha,\beta)$.

These properties already guarantee the linear independence of the standard bideterminants, for if $0 = \sum_{(U,V)\in\mathrm{SBT}(\alpha,\beta)} a_{UV}\ (U|V)$ is a non-trivial linear relation, then the finite, non-empty set $\{(U,V)|a_{UV} \neq 0\}$ has a $\mathrel{\hookleftarrow\!\Uparrow}$-maximal element (S,T). Applying C_{ST} to the last identity, using then (1), (2) and the $\mathrel{\hookleftarrow\!\Uparrow}$-maximality of (S,T), we get the following contradiction:

$$0 = \sum_{(U,V)} a_{UV} C_{ST}(U|V) = a_{ST} C_{ST}(S|T) \neq 0.$$

This shows that the linear independence of $\mathrm{SBD}(\alpha,\beta)$ is a simple consequence of (1) and (2).

Next we introduce a class of operators, which will play a dominant role in the sequel. All Capelli operators are contained in this class.

For every matrix $D = (d_{ij})$ with non-negative integral entries summing up to d, we define a left operator L_D and a right operator R_D in $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}[X]_d)$ as follows. If (U,V) is an A-bitableau, $|A| = d$, then

$$\begin{aligned} L_D\{U|V\} &:= \sum_{S \in \mathrm{Sub}_D(U)} \{S|V\} \\ \{U|V\}R_D &:= \sum_{T \in \mathrm{Sub}_D(V)} \{U|T\}, \end{aligned}$$

where

$$\mathrm{Sub}_D(U) := \{S\colon A \to \mathbf{N} | \forall i,j: \ d_{ij} = |S^{-1}\{i\} \cap U^{-1}\{j\}|\}$$

is the set of all *D-substitutes* of U. L_D (resp. R_D) is called the *left* (resp. *right*) *D-substitution*. If $\mathrm{Sub}_D(U) = \emptyset$, then $L_D\{U|V\} = \{V|U\}R_D := 0$. Less formally, for an A-tableau U, $\mathrm{Sub}_D(U)$ is the set of all A-tableaux which result from U by replacing for all i and j in a simultaneous and disjoint manner d_{ij} entries j in U by i. In particular, $\mathrm{Sub}_D(U) = \emptyset$, unless U is of content $(\alpha_1, \alpha_2, \ldots)$, where $\alpha_j = \sum_i d_{ij}$. L_D and R_D are well-defined, for if $\{U|V\} = \{W|Z\}$ for a B-bitableau (W, Z), then, by Lemma 2.1, $W = U \circ \varphi$ and $Z = V \circ \varphi$ for a suitable bijection φ. A straightforward computation then shows that $\mathrm{Sub}_D(U \circ \varphi) = \mathrm{Sub}_D(U) \circ \varphi$. The cardinality of $\mathrm{Sub}_D(U)$ is a product of multinomial coefficients:

$$|\mathrm{Sub}_D(U)| = \prod_j \binom{|U^{-1}\{j\}|}{d_{1j}, d_{2j}, \ldots}.$$

(Since the group $\prod_j \mathrm{Sym}\,(U^{-1}\{j\})$ acts transitively on $\mathrm{Sub}_D(U)$ and the stabilizer of $S \in \mathrm{Sub}_D(U)$ is isomorphic to $\prod_{i,j} \mathrm{Sym}\,(U^{-1}\{j\} \cap S^{-1}\{i\})$ our claim follows from the orbit formula.)

We next describe the action of D-substitutions on bideterminants.

Lemma 3.1 *$L_D(U|V) = \sum_S (S|V)$ and $(V|U)R_D = \sum_S (V|S)$, where both summations run over all column-injective elements S in $\mathrm{Sub}_D(U)$. Hence, if $\mathrm{Sub}_D(U)$ contains no column-injective tableau, then $L_D(U|V) = 0 = (V|U)R_D$.*

Proof. Use Lemma 2.1 (c) and (f). ▫

Most pairs (D, U) relevant for our purposes share the property that there exists at most one column-injective tableau in $\mathrm{Sub}_D(U)$. We now describe those pairs. For a tableau S let $D(S)$ denote the matrix whose (i,j)-th entry equals the number of occurences of j in the i-th row of S: $D(S)_{ij} := |\{i\} \times \mathbf{N} \cap S^{-1}\{j\}|$.

Example.

$$\text{If} \qquad S \;=\; \begin{matrix} 1133 \\ 234 \\ 4 \end{matrix} \qquad \text{then} \qquad D(S) \;=\; \begin{pmatrix} 2020 \\ 0111 \\ 0001 \end{pmatrix}.$$

□

Now if (S,T) is a bitableau then $L_{D(S)}$ (resp. $R_{D(S)}$) is called the *left* (*right*) *Capelli operator* relative to S and

$$C_{ST} \;:=\; L_{D(S)} \circ R_{D(T)}$$

is called the *Capelli operator* w.r.t. (S,T). For later use we now prove a slightly generalized version of (1). This generalization is based on the following well-known

Lemma 3.2 (Sorting Lemma) *Let λ be a diagram, U a* column-strict *λ-tableau, i.e. the entries in each column of U are strictly increasing from top to bottom. Then rearranging the entries in each row of U from left to right in non-decreasing order yields a standard λ-tableau U^{st} the* standardization *of U.*

Example.

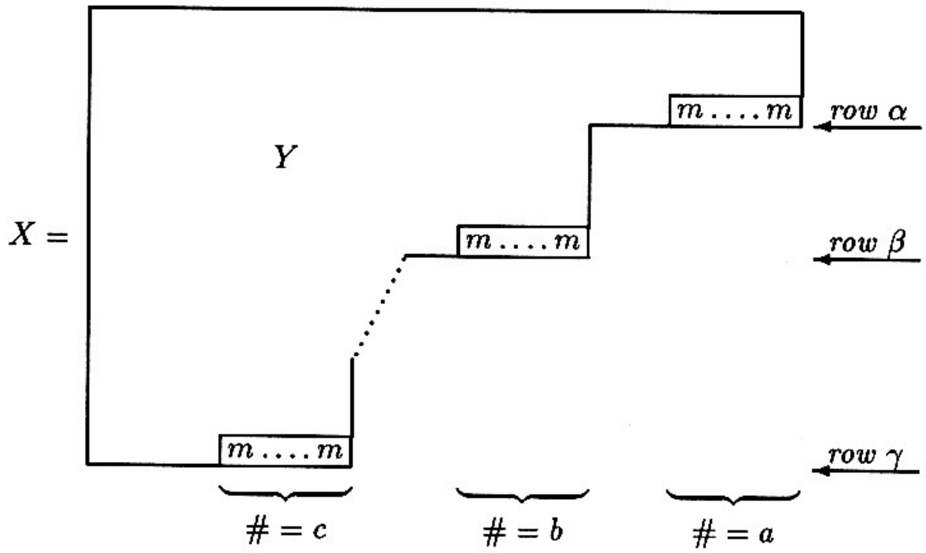
$$\text{If} \qquad U \;=\; \begin{matrix} 12113 \\ 34325 \\ 4554 \\ 5 \end{matrix} \qquad \text{then} \qquad U^{st} \;=\; \begin{matrix} 11123 \\ 23345 \\ 4455 \\ 5 \end{matrix}.$$

□

Proof. We prove the Sorting Lemma by induction on the greatest entry m in U.

$m = 1$. Since U is column-strict, we have $U \;=\; 1\ldots1 \;=\; U^{st}$.

$m > 1$. Since U is column-strict, every m in U is placed at the end of a suitable column of λ. Now ordering the columns of equal length in U according to their last entry, we get a column-strict λ-tableau X such that $X^{st} = U^{st}$. Let Y denote the "m-free part" of X:



Of course, Y is column-strict as well, and, by the maximality of m, in proceeding from X to X^{st} the m's will stay in their positions, i.e. $X \mapsto X^{st}$ is essentially described by $Y \mapsto Y^{st}$. By assumption, Y^{st} is standard, hence X^{st} $(= U^{st})$ is standard as well. $\square$

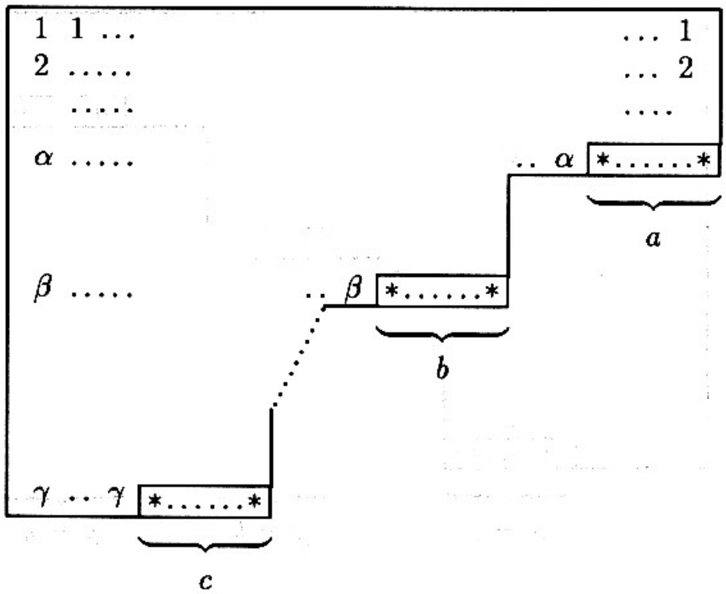
Now we can prove the first fundamental property of Capelli operators, which generalizes (1).

Theorem 3.3 *Let U and V be column-strict λ-tableaux, λ a diagram. If $P_\lambda\colon \lambda \to \mathbb{N}$ denotes the projection $(i,j) \mapsto i$, then*

$$\begin{aligned} L_{D(U)}\ (U|V) &= (P_\lambda|V) \neq 0 \\ (U|V)\ R_{D(V)} &= (U|P_\lambda) \neq 0 \\ C_{UV}\ (U|V) &= (P_\lambda|P_\lambda) \neq 0. \end{aligned}$$

Proof. By symmetry, it suffices to prove only the first statement. To begin with, note that for all horizontal permutations $\sigma \in \mathcal{H}(\lambda)$ we have $D(U) = D(U \circ \sigma)$. We refer to the notation of the previous proof.

The rearrangement of the columns of U to get X can be simulated by a suitable horizontal permutation $\sigma \in \mathcal{H}(\lambda)$: $X = U \circ \sigma$. Hence $(U|V) = (X|V \circ \sigma)$ and $L_{D(U)}\ (U|V) = L_{D(X)}\ (X|V \circ \sigma)$. If the diagram μ denotes the shape of Y, then, by induction, we can assume that the projection P_μ is the only column-injective tableau in $\mathrm{Sub}_{D(Y)}(Y)$. Hence all column-injective tableaux in $\mathrm{Sub}_{D(X)}(X)$ are necessarily of the following type



where the $a+b+\ldots+c$ $*$'s have to be replaced by a α's, b β's,..., c γ's. Obviously, P_λ is the only column-injective tableau with this property. Hence $L_{D(U)}\ (U|V) = L_{D(X)}\ (X|V\circ\sigma) = (P_\lambda|V\circ\sigma) = (P_\lambda\circ\sigma^{-1}|V) = (P_\lambda|V) \neq 0$. This completes the proof of Theorem 3.3. ◻

We now investigate the relation $\underset{\Leftarrow}{\Updownarrow}$. Tableaux $S{:}\,A\to\mathbf{N}$ and $U{:}\,B\to\mathbf{N}$ are called *row-similar* ($S\leftrightarrow U$) iff for all i the content of the i-th row of S equals the content of the same row of U. Analogously, one defines *column-similarity* ($S\updownarrow U$). Now, by definition, $S\underset{\leftarrow}{\updownarrow}U$, iff for some tableau Z, $S\leftrightarrow Z$ and $Z\updownarrow U$. If (S,T) and (U,V) are bitableaux then $(S,T)\underset{\Leftarrow}{\Updownarrow}(U,V)$ iff both $S\underset{\leftarrow}{\updownarrow}U$ and $T\underset{\leftarrow}{\updownarrow}V$.

Example.

$$\begin{array}{ccccc} & & & \begin{array}{l}11223\\3446\\5\end{array} & = U\\ & & & \updownarrow & \\ S = & \begin{array}{l}1124\\233\\45\\6\end{array} & \leftrightarrow & \begin{array}{l}1142\\3\ 2\ 3\\54\\\ \ \ 6\end{array} & \end{array}$$

hence $S\underset{\leftarrow}{\updownarrow}U$. ◻

The relation $\underset{\leftarrow}{\updownarrow}$ is reflexive but not antisymmetric. Nevertheless, $\underset{\leftarrow}{\updownarrow}$, when restricted to the set of standard tableaux, becomes antisymmetric.

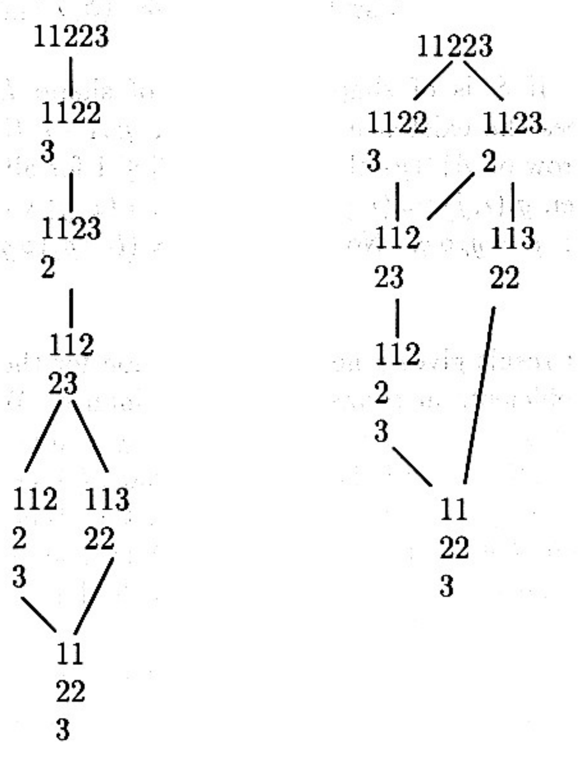
Lemma 3.4 *Let S and T be standard tableaux. Then $S = T$ iff $S \mathrel{_{\hookleftarrow}\!\!\uparrow} T$ and $T \mathrel{_{\hookleftarrow}\!\!\uparrow} S$.*

Proof. "$\Rightarrow$": trivial.

"$\Leftarrow$": For an A-tableau U and $p, q \in \mathbf{N}$ let $r_{pq}(U)$ denote the number of entries $\leq q$ in the first p rows of U. Now $S \leftrightarrow Z \updownarrow T$ implies for all p and q: $r_{pq}(S) = r_{pq}(Z) \leq r_{pq}(T)$. This combined with $T \mathrel{_{\hookleftarrow}\!\!\uparrow} S$ yields $r_{pq}(S) = r_{pq}(T)$, for all p, q. Since S and T are standard, this forces $S = T$. $\square$

The proof uses implicitly the well-known *(row) dominance* partial ordering $\trianglelefteq$ of standard tableaux: $S \trianglelefteq T$ iff $r_{pq}(S) \leq r_{pq}(T)$, for all $p, q \in \mathbf{N}$. Similarly, if $c_{pq}(U)$ denotes the number of entries $\leq q$ in the first p columns of U then, by definition, S is *column dominated* by T, $S \trianglelefteq_c T$, iff $c_{pq}(S) \leq c_{pq}(T)$ for all p, q. Let λ' denote the transpose of the diagram λ. As is well known $\lambda \trianglelefteq \mu$ iff $\mu' \trianglelefteq \lambda'$, for all diagrams λ and μ of n. Consequently, for standard tableaux S and T, both of content γ, we have $S \trianglelefteq T$ iff $T \trianglelefteq_c S$.

Obviously, $\mathrel{_{\hookleftarrow}\!\!\uparrow}$ is more restrictive than $\trianglelefteq$. The transitive closure, $\mathrel{_{\hookleftarrow}\!\!\uparrow}^*$, of $\mathrel{_{\hookleftarrow}\!\!\uparrow}$, which is a partial ordering according to Lemma 3.4, will be called *hyperdominance*. The following figure shows both the dominance and hyperdominance partial ordering for $\alpha = (2, 2, 1)$.



The following results indicate the importance of hyperbidominance.

Theorem 3.5 *If (S,T) and (U,V) are bitableaux, both of content (α,β), and if all $\alpha_i,\beta_j \in \{0,1\}$, then $C_{ST}\ (U|V) \neq 0$ iff $(S,T) \mathrel{\Leftarrow\!\!\!\Uparrow} (U,V)$, i.e. $S \mathrel{\hookleftarrow\!\!\!\uparrow} U$ and $T \mathrel{\hookleftarrow\!\!\!\uparrow} V$.*

Proof. It suffices to prove that under our assumptions, $L_{D(S)}(U|V) \neq 0$ iff $S \mathrel{\hookleftarrow\!\!\!\uparrow} U$. Since S and U are bijections with common range, $L_{D(S)}(U|V) \neq 0$ iff for all i, the entries in the i-th row of S are in different columns of U; i.e. $(\mathrm{pr}_1(S^{-1}\{j\}),\ \mathrm{pr}_2(U^{-1}\{j\})) \mapsto j$ yields a well-defined bijection Z, satisfying $S \leftrightarrow Z$ and $Z \updownarrow U$, i.e. $S \mathrel{\hookleftarrow\!\!\!\uparrow} U$. $\square$

The second fundamental property of Capelli operators, which is a generalization of (2), reads as follows.

Theorem 3.6 *For bitableaux (S,T) and (U,V), both of content (α,β), the following is valid:*

$$\begin{aligned} L_{D(S)}\ (U|V) &\neq 0 \Rightarrow\ S \mathrel{\hookleftarrow\!\!\!\uparrow} U \\ (U|V)\ R_{D(T)} &\neq 0 \Rightarrow\ T \mathrel{\hookleftarrow\!\!\!\uparrow} V \\ C_{ST}\ (U|V) &\neq 0 \Rightarrow\ (S,T) \mathrel{\Leftarrow\!\!\!\Uparrow} (U,V). \end{aligned}$$

Proof. If S is of shape A and U of shape B, then $L_{D(S)}\ (U|V) \neq 0$ guarantees the existence of a bijection $g\colon A \to B$ satisfying $U \circ g = S$ and $|g(i\text{–th row of } A) \cap y\text{–th column of } B| \leq 1$ for all i and y. Hence, if $g(i,j) = (x,y)$ then $g_r(i,j) := (i,y)$ and $g_c(i,y) := (x,y)$ yield a well-defined factorization of g: $g = g_c \circ g_r$. Now, $S = U \circ g = (U \circ g_c) \circ g_r \ \leftrightarrow U \circ g_c \updownarrow U$, i.e. $S \mathrel{\hookleftarrow\!\!\!\uparrow} U$. $\square$

The next result gives a necessary condition for the non-vanishing of straightening coefficients in terms of hyperdominance. We need some preparations. By definition, a standard tableau S is *hyperdominated* by the column-strict tableau U ($S \mathrel{\hookleftarrow\!\!\!\uparrow}^* U$) iff there exist standard (!) tableaux $S_1,\dots,S_r$ such that $S = S_1 \mathrel{\hookleftarrow\!\!\!\uparrow} S_2 \mathrel{\hookleftarrow\!\!\!\uparrow} \dots \mathrel{\hookleftarrow\!\!\!\uparrow} S_r \mathrel{\hookleftarrow\!\!\!\uparrow} U$. Similarly, the standard bitableau (S,T) is *hyperbidominated* by the column-strict bitableau (U,V) iff there exist standard bitableaux $(S_1,T_1),\dots,(S_r,T_r)$ such that $S = S_1 \mathrel{\hookleftarrow\!\!\!\uparrow} \dots \mathrel{\hookleftarrow\!\!\!\uparrow} S_r \mathrel{\hookleftarrow\!\!\!\uparrow} U$ and $T = T_1 \mathrel{\hookleftarrow\!\!\!\uparrow} \dots \mathrel{\hookleftarrow\!\!\!\uparrow} T_r \mathrel{\hookleftarrow\!\!\!\uparrow} V$. Note that hyperbidominance for standard bitableaux, denoted by $\mathrel{\Leftarrow\!\!\!\Uparrow}^*$, is more restrictive than the cartesian product of hyperdominance for standard tableaux, since hyperbidominance additionally forces $\mathrm{shape}(S_i) = \mathrm{shape}(T_i)$ for all i.

Theorem 3.7 *Let λ be a diagram, (U,V) a column-strict λ-bitableau, and let $(U|V) = \sum_{(S,T)\ \text{standard}} a_{ST,UV}\ (S|T)$. Then the following holds:*

(a) $a_{ST,UV} \neq 0 \Rightarrow\ (S,T) \hookleftarrow\!\!\!\Uparrow^{*} (U,V)$.

(b) $a_{U^{st}V^{st},UV} \ =\ 1$ *(Désarménien (1980)).*

Proof. Let (S,T) be a $\hookleftarrow\!\!\!\Uparrow$-maximal element in the *support* $\{(W,Z) | a_{WZ,UV} \neq 0\}$ of $(U|V)$. Applying the Capelli operator C_{ST} to the above formula for $(U|V)$, we get by the $\hookleftarrow\!\!\!\Uparrow$-maximality of (S,T): $C_{ST}\ (U|V) \ = \ a_{ST,UV} C_{ST}(S|T) \neq 0$. Hence $(S,T) \hookleftarrow\!\!\!\Uparrow (U,V)$. If (W,Z) lies in the support of $(U|V)$, but is not $\hookleftarrow\!\!\!\Uparrow$-maximal, then $(W,Z) \hookleftarrow\!\!\!\Uparrow^{*} (S,T)$, for some $\hookleftarrow\!\!\!\Uparrow$-maximal bitableau in that support. Thus $(W,Z) \hookleftarrow\!\!\!\Uparrow^{*} (S,T) \hookleftarrow\!\!\!\Uparrow (U,V)$, i.e. $(W,Z) \hookleftarrow\!\!\!\Uparrow^{*} (U,V)$. This proves the first statement.

The second statement results from $C_{U^{st}V^{st}}(U|V) \ = \ C_{UV}(U|V) \ = \ (P_\lambda|P_\lambda)$ and the remark that no standard tableau $W \ \neq\ U^{st}$ does exist satisfying $U^{st} \hookleftarrow\!\!\!\uparrow W \hookleftarrow\!\!\!\uparrow^{*} U$. □

Corollary 3.8 *Let U be a column-strict λ-tableau, λ a diagram. Then $(U|P_\lambda) = (U^{st}|P_\lambda) + \sum_S a_{US}(S|P_\lambda)$, for suitable $a_{US} \in \mathbf{Z}$. (The sum is over all standard λ-tableaux S which are strictly dominated by the standardization U^{st} of U.)*

4 Laplace Duality

The classical Laplace's expansions express the determinant of an n-square matrix as polynomials in certain of its minors. We generalize these results as well as Rota's straightening formula (see Doubilet *et al.* (1974), Désarménien *et al.* (1978)) by showing that rather different-looking polynomial expressions in such minors can define exactly the same polynomial in $\mathbf{Z}[X]$.

Theorem 4.1 (Laplace Duality Theorem) *Let (S,T) be an A-bitableau, $\varphi\colon A \to B$ a bijection, $\psi := \varphi^{-1}$, $(S',T') := (S \circ \psi, T \circ \psi)$. Then*

$$
\begin{aligned}
(S;\varphi|T) \ &:= \sum_{\sigma \in \mathcal{V}(B)^{\psi} \bmod \mathcal{V}(B)^{\psi} \cap \mathcal{V}(A)} \operatorname{sgn}(\sigma)(S \circ \sigma | T) \\
&= \sum_{\tau \in \mathcal{V}(A)^{\varphi} \bmod \mathcal{V}(A)^{\varphi} \cap \mathcal{V}(B)} \operatorname{sgn}(\tau)(S'|T' \circ \tau) \ =: \ (S'|T';\psi).
\end{aligned}
$$

(Here, $\mathcal{V}(B)^\psi \bmod \mathcal{V}(B)^\psi \cap \mathcal{V}(A)$ denotes an arbitrary transversal of the left cosets of $\mathcal{V}(B)^\psi \cap \mathcal{V}(A)$ in $\mathcal{V}(B)^\psi := \psi \circ \mathcal{V}(B) \circ \psi^{-1}$.)

Proof. Since both $\mathcal{V}(B)^\psi$ and $\mathcal{V}(A)$ are subgroups of $\mathrm{Sym}(A)$, we can form their complex product $\mathcal{V}(B)^\psi \cdot \mathcal{V}(A) := \{\psi \circ \beta \circ \psi^{-1} \circ \alpha | \beta \in \mathcal{V}(B), \alpha \in \mathcal{V}(A)\}$. This corresponds under inversion followed by φ-conjugation to the complex product $\mathcal{V}(A)^\varphi \cdot \mathcal{V}(B)$. Hence

$$\begin{aligned}\sum_{\sigma \in \mathcal{V}(B)^\psi \cdot \mathcal{V}(A)} \mathrm{sgn}(\sigma)\{S \circ \sigma | T\} &= \sum_{\sigma \in \mathcal{V}(B)^\psi \cdot \mathcal{V}(A)} \mathrm{sgn}(\sigma)\{S \circ \varphi^{-1} | T \circ \varphi^{-1} \circ \varphi \circ \sigma^{-1} \circ \varphi^{-1}\} \\ &= \sum_{\tau \in \mathcal{V}(A)^\varphi \cdot \mathcal{V}(B)} \mathrm{sgn}(\tau)\{S \circ \psi | T \circ \psi \circ \tau\}.\end{aligned}$$

To complete the proof, we recall that the complex product $H \cdot K$ of two subgroups H and K of a group G is a disjoint union of certain left cosets of K in G; more precisely:

$$H \cdot K = \dot{\bigcup_{h \in H \bmod H \cap K}} hK ,$$

the union is over an arbitrary transversal, $H \bmod H \cap K$, of the left cosets of $H \cap K$ in H. ▫

The classical Laplace's expansions are concerned with the case where B (or A) has exactly one non-empty column. In that case $\mathcal{V}(B) = \mathrm{Sym}(B)$ and thus $(S'|T'; \psi) = (S'|T')$. If in addition both S' and T' are of content (1^n) then, up to a sign, $(S'|T')$ equals the determinant of the generic n-square matrix (X_{ij}). Applying suitable substitutions of the form $\{X_{ij} | i, j \in \mathbb{N}\} \to \{X_{pq} | p, q \in \mathbb{N}\} \cup \{0, 1\}$ to this special case, one gets the universal determinantal identities of Abhyankar (1988).

After these remarks, we come back to the Laplace Duality Theorem. To every bijection $\psi: B \to A$ we now select a particular transversal $\mathcal{V}(B)^\psi \bmod \mathcal{V}(B)^\psi \cap \mathcal{V}(A)$. To this end we totally order the permutations of A and attach to each coset its smallest element. The total ordering of $\mathrm{Sym}(A)$ to be defined below is based on the following "twisted" total ordering $\leq_t$ of $\mathbb{N} \times \mathbb{N}$: $(a, b) \leq_t (x, y)$ iff $b > y$ or $b = y$ and $a \leq x$. Now let $A = \{a_1, \ldots, a_n\} \subset \mathbb{N} \times \mathbb{N}$, $a_1 <_t a_2 <_t \ldots <_t a_n$. We order the permutations of A $\leq_t$ -lexicographically according to their second row: $\pi <_t \sigma$ iff there exists an index i such that $\pi(a_i) <_t \sigma(a_i)$ and $\pi(a_1) = \sigma(a_1), \ldots, \pi(a_{i-1}) = \sigma(a_{i-1})$. In the sequel, if $\psi: B \to A$ is a bijection, then $\mathcal{V}(B)^\psi \bmod \mathcal{V}(B)^\psi \cap \mathcal{V}(A)$ will always denote the $\leq_t$ -lexicographically smallest transversal of the left cosets of $\mathcal{V}(B)^\psi \cap \mathcal{V}(A)$ in $\mathcal{V}(B)^\psi$. We give a more explicit characterization of those transversals. Let A^j and B^j denote the j-th columns of A and B, respectively. An easy computation then shows that

$$\begin{aligned}
&\mathcal{V}(B)^\psi \bmod \mathcal{V}(B)^\psi \cap \mathcal{V}(A) = \\
&\{\pi \in \mathcal{V}(B)^\psi | \forall j,k\colon\ \pi \text{ restricted to } \psi(B^j) \cap A^k \text{ is } \leq_t\text{-isoton}\} = \\
&\{\pi \in \mathrm{Sym}(A) | \forall j,k\colon\ \pi(\psi(B^j)) = \psi(B^j) \text{ and } \pi \downarrow (\psi(B^j) \cap A^k) \text{ is } \leq_t\text{-isoton}\}.
\end{aligned}$$

Those permutations are called *ψ-shuffles.* Note that the identity, id_A, is a ψ-shuffle for all bijections $\psi\colon B \to A$.

Example.

$$\begin{aligned}
(S;\varphi|T) \quad &:= \left(\begin{array}{ccccc|ccccc}
 & & & 5^{12} & 1^{11} & & & & 1 & 1 \\
 & & 9^{13} & 2^{21} & & & & 2 & 2 & \\
12^{14} & 10^{23} & 6^{22} & & & 3 & 3 & 3 & & \\
13^{24} & 7^{32} & 3^{31} & & & 4 & 4 & 4 & & \\
11^{33} & 8^{42} & 4^{41} & & & 5 & 5 & 5 & &
\end{array}\right) \\
&= \left(\begin{array}{cccc|cccc}
1 & 5 & 9 & 12 & 1^{15} & 1^{14} & 2^{23} & 3^{31} \\
2 & 6 & 10 & 13 & 2^{24} & 3^{33} & 3^{32} & 4^{41} \\
3 & 7 & 11 & & 4^{43} & 4^{42} & 5^{51} & \\
4 & 8 & & & 5^{53} & 5^{52} & &
\end{array}\right) =: (S'|T';\psi).
\end{aligned}$$

We illustrate the set $\mathcal{V}(B)^\psi$ mod $\mathcal{V}(B)^\psi \cap \mathcal{V}(A)$ of all ψ-shuffles, which describe the summation subject to $(S;\varphi|T)$, as follows:

$$\left\| \begin{array}{c|c|c||c|c|c||c|c|c||c}
15 & 24 & 43\ 53 & 14 & 33 & 42\ 52 & 23 & 32 & 51 & 31\ 41 \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow
\end{array} \right\|.$$

Here, elements between two adjacent double bars can arbitrarily be permuted as far as all local $\leq_t$-monotonicity conditions are satisfied. The above transversal consists of $6 \cdot 6 \cdot 6 \cdot 1$ elements; one of these shuffles is the following permutation:

$$\left\| \begin{array}{c|c|c||c|c|c||c|c|c||c}
15 & 24 & 43\ 53 & 14 & 33 & 42\ 52 & 23 & 32 & 51 & 31\ 41 \\
43 & 15 & 24\ 53 & 33 & 52 & 14\ 42 & 23 & 51 & 32 & 31\ 41
\end{array} \right\|.$$

□

5 Symmetrized Bideterminants

The horizontal group $\mathcal{H}(A)$ acts from the right via composition on the set of all A-tableaux, A a d-subset of $\mathbf{N} \times \mathbf{N}$. Symmetrized bideterminants are closely related to the $\mathcal{H}(A)$-orbits. The *left* and *right symmetrized bideterminants* relative to an A-bitableau (U, V) are defined by $[U|V) := \sum_{U' \in U \circ \mathcal{H}(A)} (U'|V)$ and $(U|V] := \sum_{V' \in V \circ \mathcal{H}(A)} (U|V')$, respectively. Symmetrized bideterminants come into the play very naturally: Let $\mathcal{L}_d$ (resp. $\mathcal{R}_d$) denote the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{Z}}(\mathbf{Z}[X]_d)$, generated by all left (resp. right) D-substitutions, where $D = (d_{ij})$ runs through all non-negative integral matrices satisfying $\sum_{ij} d_{ij} = d$. Then the following holds.

Theorem 5.1 *For every d-subset A of $\mathbf{N} \times \mathbf{N}$ and for every A-bitableau (U, V) we have*

$$\begin{aligned} \mathcal{L}_d \cdot (P_A|V) &= \sum_{S: A \to \mathbf{N}} \mathbf{Z}[S|V) \\ (U|P_A) \cdot \mathcal{R}_d &= \sum_{T: A \to \mathbf{N}} \mathbf{Z}(U|T], \end{aligned}$$

where $\mathcal{L}_d \cdot (P_A|V)$ denotes the cyclic left $\mathcal{L}_d$-module generated by $(P_A|V)$. (Recall that $P_A \colon A \to \mathbf{N}$ denotes the projection $(i,j) \mapsto i$.)

Proof. The fact that $\mathcal{L}_d \cdot (P_A|V) \supseteq \sum_S \mathbf{Z}[S|V)$ follows from the equation $[U|V) = L_{D(U)^{\mathrm{tr}}}(P_A|V)$, where $D(U)^{\mathrm{tr}}$ denotes the transpose of $D(U) = (d_{ij})$, $d_{ij} := |\{i\} \times \mathbf{N} \cap S^{-1}\{j\}|$, cf. section 3.

In order to show that $\mathcal{L}_d \cdot (P_A|V)$ is contained in $\sum_S \mathbf{Z}[S|V)$, recall that

$$\mathrm{Sub}_D(U \circ h) = \mathrm{Sub}_D(U) \circ h, \tag{3}$$

for all bijections h of A. Now let $\langle U|V) := \sum_{h \in \mathcal{H}(A)} (U \circ h|V)$. Then $\langle U|V) = |\mathrm{Stab}(U)| \cdot [U|V)$, where $\mathrm{Stab}(U)$ denotes the stabilizer of U under the $\mathcal{H}(A)$-action. Working in $\mathbf{Q}[X]$ for the moment, we have for $L_D \in \mathcal{L}_d$:

$$\begin{aligned} L_D[U|V) &= |\mathrm{Stab}(U)|^{-1} L_D \langle U|V) \\ &= |\mathrm{Stab}(U)|^{-1} \sum_{h \in \mathcal{H}(A)} \sum_{U_h \in \mathrm{Sub}_D(U \circ h)} (U_h|V) \\ &\underset{(3)}{=} |\mathrm{Stab}(U)|^{-1} \sum_{U_1 \in \mathrm{Sub}_D(U)} \sum_{h \in \mathcal{H}(A)} (U_1 \circ h|V) \\ &= \sum_{U_1 \in \mathrm{Sub}_D(U)} \frac{|\mathrm{Stab}(U_1)|}{|\mathrm{Stab}(U)|} [U_1|V). \end{aligned}$$

Now, by (3), $\mathrm{Stab}(U)$ acts on $\mathrm{Sub}_D(U)$. The stabilizer of $U_1 \in \mathrm{Sub}_D(U)$ under this $\mathrm{Stab}(U)$-action equals $\mathrm{Stab}(U) \cap \mathrm{Stab}(U_1)$. Hence

$$L_D[U|V) = \sum_{U_1 \in \mathrm{Sub}_D(U) \bmod \mathrm{Stab}(U)} [\mathrm{Stab}(U_1) : \mathrm{Stab}(U_1) \cap \mathrm{Stab}(U)]\,[U_1|V). \quad (4)$$

Finally, the index $[\mathrm{Stab}(U_1) : \mathrm{Stab}(U_1) \cap \mathrm{Stab}(U)]$ lies in $\mathbf{Z}$. □

For certain A-tableaux V, a $\mathbf{Z}$-basis of $\mathcal{L}_d \cdot (P_A|V)$ can be specified. We need some preparations. The set theoretic difference of two diagrams is called a *skew diagram*. A tableau T whose shape is a skew diagram is called a *skew tableau*. A skew tableau T is called *standard* iff the entries in each row of T are weakly increasing from left to right and the entries in each column of T are strictly increasing from top to bottom.

Example.

$$\begin{array}{cccc} & & 1 & 1 & 2 \\ & 1 & 2 & 4 & \\ 2 & & & & \\ 3 & & & & \end{array}$$

is a standard skew tableau of shape $(5,4,1,1) \setminus (2,1)$.

□

Now we can state the main result of this section, which is a variant of the well-known Gordan-Capelli formula, cf. Doubilet *et al.* (1974), Carter & Lusztig (1974), Clausen (1980a), and Barnabei & Brini (1987).

Theorem 5.2 *Let I be an injective tableau whose shape A is a skew diagram. Then the set of all left symmetrized bideterminants $[T|I)$, $T\colon A \to \mathbf{N}$ standard, is a $\mathbf{Z}$-basis of the module $[-|I) := \sum_{U:A\to\mathbf{N}} \mathbf{Z}[U|I)$.*

Proof. First we show the linear independence. Suppose, $\sum_{T \text{ standard}} a_T[T|I) = 0$ is a non-trivial linear relation. Being a finite non-empty set of standard skew tableaux of shape A, $\{T|a_T \neq 0\}$ has a $\mathrel{\leftarrow\!\!\!\uparrow}$-minimal element T_o. Now $[-|I)$ is a $\mathbf{Z}$-submodule of the space $\sum_{U:A\to\mathbf{N}} \mathbf{Z}\{U|I\}$, having all monomials $\{U|I\}, U\colon A \to \mathbf{N}$, as a $\mathbf{Z}$-basis. (Here we use the injectivity of I.) Expressing both sides of our non-trivial linear relation in terms of this basis we see that the coefficient of the monomial $\{T_o|I\}$ w.r.t. the left-hand side of the relation equals $\sum_{T \text{ standard}} a_T \sum_{(T',v)} \mathrm{sgn}(v)$, where the second summation runs over all pairs (T',v) satisfying the following conditions: $T' \in T \circ \mathcal{H}(A)$, $v \in \mathcal{V}(A)$, and $T' \circ v = T_o$. The last condition implies $T \mathrel{\leftarrow\!\!\!\uparrow} T_o$. Since both T and T_o are standard skew tableaux of shape A, the $\hookleftarrow\!\!\uparrow$-minimality of T_o forces $T = T_o$, as soon as $a_T \neq 0$. Hence the coefficient of $\{T_o|I\}$ w.r.t. the left-hand side equals a_{T_o}. Combining this with the right-hand side we get $a_{T_o} = 0$; this is a contradiction and the linear independence is proved.

We now turn to the proof that all $[T|I)$, T standard, generate $[-|I)$ as a $\mathbf{Z}$-module. Since $[U|I) = [U \circ h|I)$, for all $h \in \mathcal{H}(A)$, the module $[-|I)$ is the $\mathbf{Z}$-linear hull of all $[U|I)$, U being weakly increasing in each row. If U has this property, but is not standard, we can find the following situation in two consecutive rows of U:

$$\begin{array}{cccccccccccc}
 & a_e \leq \ldots \leq & a_{p-1} < & a_p \leq \ldots & & \ldots & & \leq a_r \\
 & \wedge \quad \ldots & \wedge & \text{\rotatebox{90}{$\leq$}} & & & & \\
b_1 \leq \ldots \leq & b_e \leq \ldots \leq & b_{p-1} \leq & b_p \leq \ldots \leq & b_q < & b_{q+1} \leq \ldots \leq b_s. & &
\end{array}$$

Here, $p := \min\{i \geq e | a_i \geq b_i\}$ and $q := \max\{j | b_j \leq a_p\}$. Note that $p > e$ implies $a_{p-1} < a_p$ since $a_{p-1} < b_{p-1} \leq b_p \leq a_p$. Now fix an $x \in \mathbf{N}$ not in the image of U; e.g. $x = 1 + \max\{u_{ij}\}$. We modify U in these two rows by replacing every a_j, $j \geq p$, and every b_l, $l \leq q$, by x. This yields an A-tableau

$$U' = \begin{array}{cccccc}
 & & & \ldots & & \\
 & a_e & \ldots & a_{p-1} & x \ldots x & x \ldots x \ldots x \\
x \ldots x & x & \ldots & x & x \ldots x & b_{q+1} \ldots b_s \\
 & & & \ldots & &
\end{array}$$

Since no tableau in $U' \circ \mathcal{H}(A)$ is column-injective, $[U'|I) = 0$. Denote the contents of $(a_p, \ldots, a_r)$, $(b_1, \ldots, b_q)$, and the x-free part of U' by α, β, and γ, respectively. Let the matrix D be defined by $D := \sum_i \{(\alpha_i + \beta_i)E_{ix} + \gamma_i E_{ii}\}$, where E_{ij} denotes the indicator matrix of the position (i,j). Using (4), a straightforward calculation then shows that the trivial identity $[U'|I) = 0$ is transformed by the left D-substitution L_D into the following identity:

$$0 \;=\; L_D[U'|I) \;=\; \sum_{\alpha',\beta'} m(\alpha',\beta')[U_{\alpha'\beta'}|I). \tag{5}$$

The summation in (5) runs over all pairs (α', β') of sequences satisfying $\alpha' + \beta' = \alpha + \beta$, $\sum_i \alpha'_i = r + 1 - p$, and $\sum_j \beta'_j = q$. $U_{\alpha'\beta'}$ results from U' by replacing the sequence of all x in the first (resp. second) relevant row of U' by the weakly increasing sequence of content α' (resp. β'). Finally,

$$m(\alpha', \beta') \;:=\; \frac{(\pi + \alpha')!}{\pi!\,\alpha'!} \cdot \frac{(\sigma + \beta')!}{\sigma!\,\beta'!} \;\in\; \mathbf{Z},$$

where $\pi := \text{content}(a_e, \ldots, a_{p-1})$, $\sigma := \text{content}(b_{q+1}, \ldots, b_s)$, and $\pi! := \pi_1!\pi_2!\ldots$. The case $\alpha' = \alpha$ and $\beta' = \beta$ is of special interest. Since $a_{p-1} < a_p < b_{q+1}$ we have in this case $(\pi + \alpha)! = \pi!\alpha!$ and $(\sigma + \beta)! = \sigma!\beta!$; hence $m(\alpha, \beta) = 1$. If $p = e$ then $\pi = (0, \ldots, 0)$, and the same reasoning shows $m(\alpha, \beta) = 1$. Altogether this allows us to rewrite formula (5) as follows:

$$[U|I) \;=\; -\sum_{(\alpha',\beta')\neq(\alpha,\beta)} m(\alpha', \beta')[U_{\alpha'\beta'}|I). \tag{6}$$

To finish the proof, it suffices to mention that U and all $U_{\alpha'\beta'}$ have the same content and all $U_{\alpha'\beta'}$ are row lexicographically smaller than U. Eq. (6) specifies the fact that a left symmetrized bideterminant $[U|I)$, which is not standard, can be written as a $\mathbf{Z}$-linear combination of smaller ones having the same content. Since there are only finitely many A-tableaux of a prescribed content, this term rewriting process terminates, and the theorem is completely proved. □

A closer look at the straightening coefficients w.r.t. the above $\mathbf{Z}$-basis of $[-|I)$ results in the following theorem.

Theorem 5.3 *Let U and I be skew tableaux of shape A, and let I be injective. Then the left symmetrized bideterminant $[U|I)$ is a $\mathbf{Z}$-linear combination of standard ones $[T|I)$, where $U \mathrel{\hookleftarrow\!\!\uparrow}^{*} T$, i.e. there exist standard skew tableaux $T_1, \ldots T_r$ of shape A such that $U \mathrel{\hookleftarrow\!\!\uparrow} T_1 \mathrel{\hookleftarrow\!\!\uparrow} \ldots \mathrel{\hookleftarrow\!\!\uparrow} T_r = T$:*

$$[U|I) \;\in\; \sum_{T \text{ standard: } U \mathrel{\hookleftarrow\!\!\uparrow}^{*} T} \mathbf{Z}[T|I).$$

Proof. According to the last theorem,

$$[U|I) \;=\; \sum_{T \text{ standard}} a_T[T|I), \tag{7}$$

for suitable $a_T \in \mathbf{Z}$. Let T_o be a $\hookleftarrow\!\!\uparrow$-minimal element in the support $\{T|a_T \neq 0\}$. Comparing the coefficients of the monomial $\{T_o|I\}$ in (7), we get in analogy to the last proof

$$\sum_{(U',v)} \text{sgn}(v) \;=\; a_{T_o} \;\neq\; 0,$$

where the summation is over all pairs (U', v) such that $U' \in U \circ \mathcal{H}(A)$ and $v \in \mathcal{V}(A)$ satisfy $U' \circ v = T_o$. Hence $U \mathrel{\hookleftarrow\!\!\uparrow} T_o$; since T_o is $\hookleftarrow\!\!\uparrow$-minimal, our claim follows. □

Without proof we mention the following result.

Theorem 5.4 *The left (respectively right) symmetrized bideterminants corresponding to all standard bitableaux form a* $\mathbf{Q}$*-basis of* $\mathbf{Q}[X]$.

6 Adjointness

Ordinary and symmetrized bideterminants are adjoint to each other. The goal of the present section is to make this more precise.

We first recall some notions and facts from algebra. Let V, W be $\mathbf{Z}$-modules. A symmetric $\mathbf{Z}$-bilinear form $f: V \times V \to \mathbf{Z}$ is called *non-singular* iff $x \mapsto (y \mapsto f(x,y))$ defines an isomorphism $V \to \mathrm{Hom}_{\mathbf{Z}}(V, \mathbf{Z})$ of $\mathbf{Z}$-modules. If both $f: V \times V \to \mathbf{Z}$ and $g: W \times W \to \mathbf{Z}$ are non-singular, then to every $C \in \mathrm{Hom}_{\mathbf{Z}}(V, W)$ there exists one and only one $C^* \in \mathrm{Hom}_{\mathbf{Z}}(V, W)$ satisfying $g \circ (C \times \mathrm{id}_W) = f \circ (\mathrm{id}_V \times C^*)$. C and C^* are called (f,g)*-adjoint*.

Now we apply this to our situation. For a fixed finite subset A of $\mathbf{N} \times \mathbf{N}$ let $I_A: A \to \mathbf{N}$ be any injection, and let $P_A: A \to \mathbf{N}$ denote the projection $(i,j) \mapsto i$. We summarize some crucial facts relating the $\mathbf{Z}$-submodules

$$\{-|P_A\} := \sum_{S:A \to \mathbf{N}} \mathbf{Z}\{S|P_A\} \quad \text{and} \quad (-|I_A) := \sum_{S:A \to \mathbf{N}} \mathbf{Z}(S|I_A).$$

Theorem 6.1

(a) $\{-|P_A\}$ *is a free* $\mathbf{Z}$*-module. The monomials* $\{S|P_A\}$, $S: A \to \mathbf{N}$ *weakly increasing in each row, form a* $\mathbf{Z}$*-basis of* $\{-|P_A\}$.

(b) $(-|I_A)$ *is a free* $\mathbf{Z}$*-module. The bideterminants* $(S|I_A)$, $S: A \to \mathbf{N}$ *strictly increasing in each column, are a* $\mathbf{Z}$*-basis of* $(-|I_A)$.

(c) $(S|I_A) \mapsto (S|P_A)$ *defines a* $\mathbf{Z}$*-linear mapping* $C_A: (-|I_A) \to \{-|P_A\}$. *(In fact,* C_A *is the restriction to* $(-|I_A)$ *of the right Capelli operator w.r.t.* I_A.*)*

(d) $\{S|P_A\} \mapsto [S|I_A)$ *defines a* $\mathbf{Z}$*-linear mapping* $C_A^*: \{-|P_A\} \to (-|I_A)$.

(e) *Viewing the* $\mathbf{Z}$*-bases specified in (a) and (b) as orthonormal bases with corresponding* $\mathbf{Z}$*-bilinear forms* $\mathcal{P}: \{-|P_A\}^2 \to \mathbf{Z}$ *and* $\mathcal{I}: (-|I_A)^2 \to \mathbf{Z}$, C_A *and* C_A^* *are* $(\mathcal{P}, \mathcal{I})$*-adjoint. More precisely, for all* $S, S': A \to \mathbf{N}$ *we have*

$$\begin{aligned} \mathcal{P}\left((S|P_A),\ \{S'|P_A\}\right) &= \sum_{\sigma \in \mathcal{V}(A),\ S \circ \sigma \in S' \circ \mathcal{H}(A)} \mathrm{sgn}(\sigma) \\ &= \mathcal{I}\left((S|I_A),\ [S'|I_A)\right). \end{aligned}$$

Proof. The case that A is a skew diagram is proved by Barnabei & Brini (1987). The results easily generalize to arbitrary A. $\square$

The images

$$\operatorname{im}(C_A) = \sum_{S:A\to\mathbf{N}} \mathbf{Z}(S|P_A) \quad \text{and} \quad \operatorname{im}(C_A^*) = \sum_{S:A\to\mathbf{N}} \mathbf{Z}[S|I_A)$$

have been called *Schur* and *Co-Schur module* respectively, cf. Barnabei & Brini (1987). By Theorem 5.1,

$$\operatorname{im}(C_A^*) = \mathcal{L}_d \cdot (P_A|I_A).$$

Restricting $\operatorname{im}(C_A)$ and $\operatorname{im}(C_A^*)$ to that part corresponding to all A-tableaux of a prescribed content α we get the following local version of the above results.

Corollary 6.2

$$\sum_{\substack{S:A\to\mathbf{N}\\ \text{content}(S)=\alpha}} \mathbf{Z}(S|P_A) \quad \textit{and} \quad \sum_{\substack{S:A\to\mathbf{N}\\ \text{content}(S)=\alpha}} \mathbf{Z}[S|I_A)$$

are finitely generated free $\mathbf{Z}$*-modules of equal rank.*

Proof. Recall that submodules of free $\mathbf{Z}$-modules are free as well. Now the rank of both modules is the rank of the matrix $(\mathcal{P}((S|P_A),\ \{S'|P_A\})) = (\mathcal{I}((S|I_A),\ [S'|I_A)))$, where S (resp. S') ranges over all A-tableaux of content α, which are strictly (resp. weakly) increasing in each column (resp. row). $\square$

7 Group Representations

The following sections are concerned with applications to representation theory. To keep this paper self-contained to some extent, the present section recalls fundamental notions, facts and problems of the theory of group representations.

Let G be a group, F a field. An *F-representation* of G of *degree d* with *representation space V* is a group morphism $D: G \to GL(V)$, where V is a d-dimensional vector space over F. Choosing an F-basis of V, every F-automorphism $D(g)$, $g \in G$, is described by an invertible d-square matrix $\mathbf{D}(g)$ over F, and $g \mapsto \mathbf{D}(g)$ is a group morphism $\mathbf{D}: G \to GL(d, F)$, a so-called *matrix representation* of G. Typically, different F-bases of V will give rise to different but "equivalent" matrix representations corresponding to D.

More generally, two F-representations $D_i:\ G \to GL(V_i)\ \ (i = 1,2)$ are called *equivalent*, $D_1 \sim D_2$, iff there exists an F-isomorphism $T:\ V_1 \to V_2$ such that $D_2(g) = T \circ D_1(g) \circ T^{-1}$, for all $g \in G$. A central problem in representation theory is the classification of all F-representations of G up to equivalence. For the moment, let us assume that G is a finite group. The classification problem substantially depends on how the order of G and the characteristic of F are related. There are two alternatives leading to completely different theories: *ordinary representation theory* (char F does not divide $|G|$) and *modular representation theory* (char F divides $|G|$). We sketch the ordinary theory, although the methods presented so far also have applications to the modular theory, cf. Clausen (1979,1980b), Green (1980), Golembiowski (1987), Pittaluga & Strickland (1988). In the case of the ordinary representation theory it is allowed to take averages of the form $|G|^{-1}\sum_{g\in G}$. With the help of such averages one can prove the following.

Theorem 7.1 *(Maschke). If the characteristic of the field F does not divide the order of the finite group G, then every F-representation $D:\ G \to GL(V)$ is a direct sum of irreducible F-representations:*

$$D = D_1 \oplus \ldots \oplus D_s, \quad D_i \textit{ irreducible.}$$

That is to say, $V = V_1 \oplus \ldots \oplus V_s$ is the direct sum of F-subspaces $V_i \neq 0$, each V_i is *G-invariant* (i.e. $D(g)V_i \subseteq V_i$, $\forall g \in G$) and in addition, besides 0 and V_i there are no further G-invariant subspaces in V_i. The restriction of D to V_i yields an *irreducible representation* $D_i:\ G \to GL(V_i)$. If $D = d_1 \oplus \ldots \oplus d_t$ is another decomposition of D into irreducible constituents d_i, then a result of Krull-Remak-Schmidt guarantees $s = t$ and (after a suitable permutation) the equivalence of D_i and d_i $(i = 1,\ldots,s)$. Consequently, if Δ is an irreducible F-representation of G then the *multiplicity* $\langle\Delta|D\rangle := |\{i|D_i \sim \Delta\}|$ of Δ in D is well-defined. Thus in case of the ordinary representation theory the classification problem splits as follow:

(i) Compute a transversal $\mathrm{Irrep}(G,F)$ of the equivalence classes of irreducible F-representations of G.

(ii) Given any F-representation D of G, determine its equivalence type by computing all the multiplicities $\langle\Delta|D\rangle$, $D \in \mathrm{Irrep}(G,F)$.

In connection with (i) the following questions naturally arise: How many equivalence classes of irreducible F-representations do exist? Where to look for irreducible representations? At least theoretically, it is easy to answer these questions: The F-vector space FG with F-basis G is the representation space of G via $R(g)(x) := gx$ $(g, x \in G)$. Every irreducible F-representation of G is equivalent to an irreducible constituent of this so-called *regular F-representation* $R: G \to GL(FG)$ of G. It turns out that FG (via group multiplication) is a semisimple F-algebra. Applying Wedderburn's theory of semisimple algebras to this special situation results in the following classical theorem.

Theorem 7.2 *Let F be a field whose characteristic does not divide the order of the finite group G. Let h denote the number of conjugacy classes of G and let R denote the regular F-representation of G. Then*

(a) $|\mathrm{Irrep}(G, F)| \leq h$;

(b) *for all* $\Delta \in \mathrm{Irrep}(G, F)$, $1 \leq \langle \Delta | R \rangle \leq \mathrm{degree}(\Delta)$.

(c) *If F contains in addition a primitive $|G|$-th root of unity then there are exactly h classes of irreducible F-representations of G and* $\langle \Delta | R \rangle = \mathrm{degree}(\Delta)$, *for all* $\Delta \in \mathrm{Irrep}(G, F)$.

In the next section we will present transversals for irreducible F-representations of symmetric groups, $G = S_n$. Section 8 is concerned with the computation of the equivalence classes of certain reducible representations $D: G \to GL(V)$. To this end it suffices to compute a *composition series* of V, i.e. a maximal chain $0 = U_0 \subset U_1 \subset \ldots \subset U_s = V$ of G-invariant subspaces U_i of V.

In the sequel we prefer the language of module theory: If $D: G \to GL(V)$ is an F-representation of G then $gv := D(g)v$ makes V into a left FG-module. The G-invariant F-subspaces of V are called FG-submodules. V is a *simple* FG-module, iff $V \neq 0$ and V and 0 are the only FG-submodules of V. Simple modules and irreducible representations correspond to each other. An FG-module V is *cyclic* iff $V = FGv$, for some $v \in V$. A *morphism* of left FG-modules V_1, V_2 is an F-linear map $f: V_1 \to V_2$ commuting with the G-actions: $f(gv) = gf(v)$ $(g \in G, v \in V_1)$.

8 Simple Modules

This section discusses composition series for every (split) semisimple group algebra RS_n, viewed as left RS_n-module. Section 9 generalizes this to a wider class of cyclic left RS_n-modules.

Let R be a commutative ring with unit element $1 = 1_R \neq 0$. The symmetric group S_n acts as a group of algebra automorphisms on $R[X]$ via $\sigma X_{ij} := X_{\sigma i, j}$ $(\sigma i := i$ for $i > n)$. The map $\sigma \mapsto \prod_{1 \leq i \leq n} X_{\sigma i, i}$ yields an isomorphism $RS_n \to R_{\alpha\beta}$ of left RS_n-modules, where $\alpha = \beta = (1^n)$, cf. Theorem 2.2.

For an A-bitableau (U, V), a permutation $\sigma \in S_n$, and a right substitution R_D satisfying $\sum_{i,j} d_{i,j} = |A|$ one easily checks that $\sigma\{U|V\} = \{\sigma \circ U|V\}$, $\sigma(U|V) = (\sigma \circ U|V)$, $(\sigma\{U|V\})R_D = \sigma(\{U|V\}R_D)$. In particular, the right Capelli operators are RS_n-morphisms.

According to the local version of Theorem 2.2, the standard bideterminants of content $((1^n), (1^n))$ form an R-basis of $R_{(1^n)(1^n)}$. A suitable arrangement of this basis indicates a close relationship to Wedderburn's structure theory of (split) semisimple (group) algebras: Let $T_1 \leq \ldots \leq T_r$ be a total ordering of all r standard tableaux of content (1^n) such that $T_i \lrcorner\!\!\uparrow^* T_j \Rightarrow T_i \leq T_j$, for all i, j. W.r.t. this ordering we define a partial r-square matrix $(b_{ij}^{(n)})$, which lists all elements of $\mathrm{SBD}(1^n, 1^n)$:

$$b_{ij}^{(n)} := \begin{cases} (T_i|T_j)\ , & \text{if shape}(T_i) = \text{shape}(T_j) \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

The matrix $(b_{ij}^{(4)})$, based on the column lexicographical ordering of all standard tableaux of content (1^4), reads as follows:

$$\left(\begin{array}{l|l} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{array}\right)$$

$$\begin{array}{ccc}
\left(\begin{array}{l|l} 14 & 14 \\ 2 & 2 \\ 3 & 3 \end{array}\right) & \left(\begin{array}{l|l} 14 & 13 \\ 2 & 2 \\ 3 & 4 \end{array}\right) & \left(\begin{array}{l|l} 14 & 12 \\ 2 & 3 \\ 3 & 4 \end{array}\right) \\
\left(\begin{array}{l|l} 13 & 14 \\ 2 & 2 \\ 4 & 3 \end{array}\right) & \left(\begin{array}{l|l} 13 & 13 \\ 2 & 2 \\ 4 & 4 \end{array}\right) & \left(\begin{array}{l|l} 13 & 12 \\ 2 & 3 \\ 4 & 4 \end{array}\right) \\
\left(\begin{array}{l|l} 12 & 14 \\ 3 & 2 \\ 4 & 3 \end{array}\right) & \left(\begin{array}{l|l} 12 & 13 \\ 3 & 2 \\ 4 & 4 \end{array}\right) & \left(\begin{array}{l|l} 12 & 12 \\ 3 & 3 \\ 4 & 4 \end{array}\right)
\end{array}$$

$$\begin{array}{cc}
\left(\begin{array}{l|l} 13 & 13 \\ 24 & 24 \end{array}\right) & \left(\begin{array}{l|l} 13 & 12 \\ 24 & 34 \end{array}\right) \\
\left(\begin{array}{l|l} 13 & 13 \\ 24 & 24 \end{array}\right) & \left(\begin{array}{l|l} 13 & 12 \\ 24 & 34 \end{array}\right)
\end{array}$$

$$\begin{array}{ccc}
\left(\begin{array}{l|l} 134 & 134 \\ 2 & 2 \end{array}\right) & \left(\begin{array}{l|l} 134 & 124 \\ 2 & 3 \end{array}\right) & \left(\begin{array}{l|l} 134 & 123 \\ 2 & 4 \end{array}\right) \\
\left(\begin{array}{l|l} 124 & 134 \\ 3 & 2 \end{array}\right) & \left(\begin{array}{l|l} 124 & 124 \\ 3 & 3 \end{array}\right) & \left(\begin{array}{l|l} 124 & 123 \\ 3 & 4 \end{array}\right) \\
\left(\begin{array}{l|l} 123 & 134 \\ 4 & 2 \end{array}\right) & \left(\begin{array}{l|l} 123 & 124 \\ 4 & 3 \end{array}\right) & \left(\begin{array}{l|l} 123 & 123 \\ 4 & 4 \end{array}\right)
\end{array}$$

$$(1234|1234)$$

Let $U_{\rho,n}(R)$ denote the R-linear hull of all bideterminants in the first ρ columns of $(b_{ij}^{(n)})$.

Theorem 8.1 *Let r be the number of standard tableaux of content (1^n) and let R be a commutative ring with $1_R \neq 0$. Then, referring to the above notations, the following holds:*

(a) $0 =: U_{0,n}(R) \subset U_{1,n}(R) \subset \ldots \subset U_{r,n}(R) \simeq RS_n$ *is a chain of RS_n-submodules of $R_{(1^n)(1^n)}$.*

(b) *The RS_n-morphism $R_{D(T_\rho)}$ $(1 \leq \rho \leq r)$ maps $U_{\rho,n}(R)$ onto the so-called* Specht module

$$\mathcal{S}_\lambda(R) := RS_n \cdot (S|P_\lambda) = \langle\langle (T|P_\lambda) | T \in ST^\lambda(1^n) \rangle\rangle_R.$$

Here, T_ρ is assumed to be of shape λ and $S : \lambda \to \{1, \ldots, n\}$ is any bijection.

(c) $U_{\rho-1,n}(R) = \mathrm{Kernel}(R_{D(T_\rho)} \downarrow U_{\rho,n}(R))$, *for every ρ, $1 \leq \rho \leq r$. Hence the chain $U_{0,n}(R) \subset \ldots \subset U_{r,n}(R)$ is a* Specht series *of RS_n, i.e. all factors corresponding to this chain are isomorphic to Specht modules.*

(d) *If R is a field whose characteristic does not divide $n!$ then the chain in (b) is a composition series of the left regular module RS_n. Furthermore,*

$$\{\mathcal{S}_\lambda(R) | \lambda \text{ a partition of } n\}$$

is a transversal of the isomorphism classes of simple left RS_n-modules.

Proof.

(a). Let $b_{ij}^{(n)} \in U_{\rho,n}(R)$ and let $\sigma \in S_n$. Then, by definition, $i \leq \rho$ and $\sigma b_{ij}^{(n)} = \sigma(T_i|T_j) = (\sigma \circ T_i|T_j)$ is a $\mathbf{Z}1_R$-linear combination of standard bideterminants $(T_p|T_q)$, such that $T_q \hookleftarrow^* T_j$. Hence $T_q \leq T_j$ i.e. $q \leq j \leq \rho$. This proves (a).

(b) and (c). Let T_ρ be of shape λ. By the first fundamental property of Capelli operators (Theorem 3.3) we get for all T_p of shape λ:

$$(T_p|T_\rho)R_{D(T_\rho)} = (T_p|T_\lambda) \; .$$

Hence $\mathcal{S}_\lambda(R)$ is contained in the image of $R_{D(T_\rho)} \downarrow U_{\rho,n}(R)$. On the other hand let $b_{ij}^{(n)} \in U_{\rho,n}(R), j < \rho$. As the total ordering $\leq$ is a linearization of $\hookleftarrow^*$, it is impossible that $T_\rho \hookleftarrow^* T_j$ holds. By the second fundamental property of Capelli operators (Theorem 3.6), this yields

$$b_{ij}^{(n)} R_{D(T_\rho)} = 0 \; .$$

Combining this with the linear independence of $\{(T|T_\lambda) | T \in ST^\lambda(1^n)\}$, we get altogether

$$\begin{aligned} \mathcal{S}_\lambda(R) &= \mathrm{Image}(R_{D(T_\rho)} \downarrow U_{\rho,n}(R)) \\ U_{\rho-1,n}(R) &= \mathrm{Kernel}(R_{D(T_\rho)} \downarrow U_{\rho,n}(R)). \end{aligned}$$

(d). First we observe that the Specht module $\mathcal{S}_\lambda(R)$ occurs in $U_{0,n} \subset .. \subset U_{r,n}(R)$ at least $\dim_R \mathcal{S}_\lambda(R)$ times. Under the assumption, RS_n is a semisimple R-algebra (by Maschke's Theorem). In general, if A is a semisimple algebra over the field R and M is a simple left A-module, then (by Wedderburn's Theorem) for the multiplicity $\langle M|A\rangle$ of M in any composition series of A (as a left A-module) the following holds:

$$1 \leq \langle M|A\rangle \leq \dim_R(M) .$$

Now (d) follows easily. □

Example. If $R = \mathbf{Q}$ is the field of rational numbers and $n = 4$, a transversal of simple $\mathbf{Q}S_4$-modules is given by the following list of Specht modules:

$$\mathcal{S}_{(4)}(\mathbf{Q}) = \langle\!\langle \begin{pmatrix} 1\,2\,3\,4 & \!\!\!\big|\, 1\,1\,1\,1 \end{pmatrix} \rangle\!\rangle_{\mathbf{Q}},$$

$$\mathcal{S}_{(3,1)}(\mathbf{Q}) = \langle\!\langle \left(\begin{array}{lll|lll} 1 & 3 & 4 & 1 & 1 & 1 \\ 2 & & & 2 & & \end{array}\right), \left(\begin{array}{lll|lll} 1 & 2 & 4 & 1 & 1 & 1 \\ 3 & & & 2 & & \end{array}\right), \left(\begin{array}{lll|lll} 1 & 2 & 3 & 1 & 1 & 1 \\ 4 & & & 2 & & \end{array}\right) \rangle\!\rangle_{\mathbf{Q}},$$

$$\mathcal{S}_{(2,2)}(\mathbf{Q}) = \langle\!\langle \left(\begin{array}{ll|ll} 1 & 3 & 1 & 1 \\ 2 & 4 & 2 & 2 \end{array}\right), \left(\begin{array}{ll|ll} 1 & 2 & 1 & 1 \\ 3 & 4 & 2 & 2 \end{array}\right) \rangle\!\rangle_{\mathbf{Q}},$$

$$\mathcal{S}_{(2,1,1)}(\mathbf{Q}) = \langle\!\langle \left(\begin{array}{ll|ll} 1 & 4 & 1 & 1 \\ 2 & & 2 & \\ 3 & & 3 & \end{array}\right), \left(\begin{array}{ll|ll} 1 & 3 & 1 & 1 \\ 2 & & 2 & \\ 4 & & 3 & \end{array}\right), \left(\begin{array}{ll|ll} 1 & 2 & 1 & 1 \\ 3 & & 2 & \\ 4 & & 3 & \end{array}\right) \rangle\!\rangle_{\mathbf{Q}},$$

$$\mathcal{S}_{(1,1,1,1)}(\mathbf{Q}) = \langle\!\langle \left(\begin{array}{l|l} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{array}\right) \rangle\!\rangle_{\mathbf{Q}} .$$

9 Skew Modules

In this section we associate to every n-subset A of $\mathbf{N} \times \mathbf{N}$ a cyclic (left) RS_n-module $\mathcal{S}_A(R)$. If A is a skew diagram, then $\mathcal{S}_A(R)$ is called a *skew module*, if A is a diagram, $\mathcal{S}_A(R)$ specializes to the Specht module corresponding to A. For every skew diagram A, we will describe two R-bases of the skew module $\mathcal{S}_A(R)$. The first basis corresponds to all standard A-tableaux of content (1^n), whereas the second one, adapted to a Specht series of $\mathcal{S}_A(R)$, corresponds to certain standard bitableaux. The next section presents an algorithm which efficiently generates such R-bases adapted to Specht series.

To begin with let A be an n-subset of $\mathbb{N} \times \mathbb{N}$ and let P_A denote the projection $A \ni (i,j) \mapsto i$. Then, by definition, $\mathcal{S}_A(R)$ is the R-linear hull of all bideterminants $(U|P_A)$, where U runs through all bijections $A \to \{1,\ldots,n\}$, i. e. U is an A-tableau of content (1^n). Since $\sigma(U|P_A) = (\sigma \circ U|P_A)$ for all $\sigma \in S_n$, $\mathcal{S}_A(R)$ is a cyclic RS_n-module generated by $(U|P_A)$, where U is any fixed A-tableau of content (1^n):

$$\mathcal{S}_A(R) = RS_n \cdot (U|P_A) .$$

In the sequel we mainly discuss the case when A is a skew diagram. Our first goal is to describe an R-basis of $\mathcal{S}_A(R)$ closely related to A, see James (1977), James & Peel (1979) and Clausen (1980a).

Theorem 9.1 *Let A be a skew diagram. Then the set of all bideterminants $(U|P_A)$, where U runs through all standard A-tableaux of content (1^n), forms an R-basis of $\mathcal{S}_A(R)$.*

Proof. Linear independence: Let $\sum_{U \text{ standard}} a_U(U|P_A) = 0$ be a non-trivial linear relation and let S be $\lrcorner\!\!\uparrow^*$-maximal in $\{U|a_U \neq 0\}$. Then $L_{D(S)}(S|P_A) = (P_A|P_A)$ and, by Theorem 2.6, $L_{D(S)}(U|P_A) = 0$ for all $U \neq S$ satisfying $a_U \neq 0$. Equating coefficients we get $0 \neq a_S = 0$, a contradiction.

Span: Use the Laplace Duality Theorem 4.1 and a "transposed" version of the proof of Theorem 5.2. The details are left to the reader. $\square$

Now we are going to describe a second R-basis of the skew module $\mathcal{S}_A(R)$ which is adapted to a Specht series. By Theorem 2.2 every $X \in \mathcal{S}_A(R)$ can uniquely be written as an R-linear combination of standard bideterminants:

$$X = \sum_{(U,V) \text{ standard}} \xi_{U,V}(U|V) .$$

We will call

$$\operatorname{supp}_r(X) := \{V | \exists U : \xi_{U,V} \neq 0\} \quad \text{and} \quad \operatorname{supp}_r(M) := \bigcup_{X \in M} \operatorname{supp}_r(X)$$

the *right support* of $X \in R[X_{ij}]$ and $M \subseteq R[X_{ij}]$, respectively. The following theorem indicates the importance of this notion.

Theorem 9.2 *Let A be any n-subset of $\mathbb{N} \times \mathbb{N}$ and let λ be a partition of n. Furthermore let R be a field of characteristic zero and let $\leq$ be any partial ordering on the set of all standard tableaux refining $\lrcorner\!\!\uparrow^*$. Then for the multiplicity of the Specht module $\mathcal{S}_\lambda(R)$ in $\mathcal{S}_A(R)$ the following holds:*

(a) $\langle \mathcal{S}_\lambda(R)|\mathcal{S}_A(R)\rangle \leq |\{T|T$ *is a* $\leq$*-maximal element in* $\mathrm{supp}_r(X)$ *for some* $X \in \mathcal{S}_A(R)$ *and shape*$(T) = \lambda\}|$.

(b) $\langle \mathcal{S}_\lambda(R)|\mathcal{S}_A(R)\rangle \geq |\{T|T$ *is the* $\leq$*-greatest element in* $\mathrm{supp}_r(X)$ *for some* $X \in \mathcal{S}_A(R)$ *and shape*$(T) = \lambda\}|$.

Proof. The right support of $M := \mathcal{S}_A(R)$ is finite since every $T \in \mathrm{supp}_r(M)$ is a standard tableau whose content equals the content of P_A. Thus if $\mathrm{supp}_r(M)$ consists of the r standard tableaux $T_1, \ldots, T_r$ then we may arrange the T_i in such a way that

$$T_i \hookleftarrow\!\!\!\rceil^* T_j \Rightarrow T_i \leq T_j \Rightarrow i \leq j .$$

Let $M_i := \{X \in M|\mathrm{supp}_r(X) \subseteq \{T_1, \ldots, T_i\}\}$. Then, by Theorem 3.7, $0 = M_0 \leq M_1 \leq \ldots \leq M_r = M$ is a chain of RS_n-submodules of M. Moreover, if M_i/M_{i-1} is non-zero and T_i is of shape λ, then M_i/M_{i-1} is isomorphic to the Specht module $\mathcal{S}_\lambda(R)$. In order to see this let $X \in M_i \backslash M_{i-1}$. Then $X = X_1 + X_2 + \ldots + X_i$, where X_j is a linear combination of standard bideterminants $(U|T_j)$, U a standard tableau of content (1^n). According to our assumption, X_i is non-zero. Furthermore, the right Capelli operator $R_{D(T_i)}$, which is an RS_n-morphism, maps X_i onto a non-zero element of the Specht module $\mathcal{S}_\lambda(R)$ whereas, by Theorem 3.6, $R_{D(T_i)}$ annihilates every X_j, for $1 \leq j < i$. Thus M_{i-1} is in the kernel of $R_{D(T_i)}$ whereas M_i is mapped by $R_{D(T_i)}$ onto a non-zero RS_n-submodule of $\mathcal{S}_\lambda(R)$. By the simplicity of the Specht modules, $R_{D(T_i)}(M_i) = \mathcal{S}_\lambda(R)$. Finally, since $\dim_R(M_i/M_{i-1}) \leq \dim_R(\mathcal{S}_\lambda(R))$, a dimension argument shows that M_i/M_{i-1} and $\mathcal{S}_\lambda(R)$ are isomorphic RS_n-modules. Hence

$$\langle \mathcal{S}_\lambda(R)|\mathcal{S}_A(R)\rangle = |\{i|M_i > M_{i-1} \text{ and } \mathrm{shape}(T_i) = \lambda\}| .$$

Now $M_i > M_{i-1}$ implies that T_i is a $\leq$-maximal element in $\mathrm{supp}_r(X)$ for every $X \in M_i \backslash M_{i-1}$. This proves statement (a). (Note that in general $\mathrm{supp}_r(X)$ has several $\leq$-maximal elements. But only one of it contributes to the multiplicities.) If T_i is the $\leq$-greatest element in $\mathrm{supp}_r(X)$ then $M_i > M_{i-1}$ and $M_i/M_{i-1} \simeq \mathcal{S}_\lambda(R)$, where λ is the shape of T_i. This proves our second statement. □

In the sequel we will show that if $\leq$ is the dominance partial ordering $\trianglelefteq$ of standard tableaux and if A is a skew diagram then

$$\{T|T \text{ is a } \trianglelefteq -\text{maximal element in } \mathrm{supp}_r(X) \text{ for some } X \in \mathcal{S}_A(R)\} =$$
$$\{T|T \text{ is the } \trianglelefteq -\text{greatest element in } \mathrm{supp}_r(X) \text{ for some } X \in \mathcal{S}_A(R)\} .$$

In addition, Theorem 9.4 characterizes those T by a suitable combinatorial condition. We need some preparations.

Let A and B be two n-subsets of $\mathbf{N} \times \mathbf{N}$. Furthermore, let α_i and β_i denote the length of the ith row of A and B, respectively. Then $\alpha = (\alpha_1, \alpha_2, \ldots)$

(resp. $\beta = (\beta_1, \beta_2, \ldots)$) is the content of the projection P_A (resp. P_B) and $\sum_i \alpha_i = \sum_j \beta_j = n$. Let $T^{A,\leq}(\beta)$ denote the set of all A-tableaux of content β that are weakly increasing from left to right in each row of A. Define $T^{B,\leq}(\alpha)$ in a similar way. It is easy to see that there is a unique bijection

$$\begin{array}{cccc} * \; : & T^{A,\leq}(\beta) & \rightarrow & T^{B,\leq}(\alpha) \\ & T = (t_{ip}) & \mapsto & T^* = (t^*_{jq}) \end{array}$$

satisfying for all i and j

$$|\{p | t_{ip} = j\}| = |\{q | t^*_{jq} = i\}| \; .$$

In a sense, this bijection dualizes content and shape. Therefore T^* will be called the B-*dual* of T. Interchanging the rôles of A and B, every $S \in T^{B,\leq}(\alpha)$ is associated with its A-dual S^*. The A-dual of the B-dual T^* of $T \in T^{A,\leq}(\beta)$ equals T: $T^{**} = T$.

In our applications, A will be a fixed skew diagram and B varies through all partitions of n. The bijections just mentioned will help to link the set $ST^A(\beta)$ of all standard skew tableaux of shape A and content β and the set $ST^B(\alpha)$ of all standard tableaux of shape B and content α. One of our goals is to prove that

$$|ST^B(\alpha)^* \cap ST^A(\beta)|$$

is the multiplicity with which the simple module $\mathcal{S}_B(R)$ occurs in the skew module $\mathcal{S}_A(R)$. (If $\mathcal{T} \subseteq T^{B,\leq}(\alpha)$ then $\mathcal{T}^*$ will denote the set of all T^*, $T \in \mathcal{T}$.)

Example Let

$$A = \begin{array}{cccc} & & & \times \\ & \times & \times & \times \\ \times & \times & \times & \\ \times & \times & & \end{array} \quad \text{and} \quad B = \begin{array}{cccc} \times & \times & \times & \times \\ \times & \times & & \\ \times & \times & & \\ \times & & & \end{array} \; .$$

Then $\alpha = (1,3,3,2)$ and $\beta = (4,2,2,1)$. We show two standard tableaux T, U of content α and its A-duals:

$$T^* = \begin{array}{cccc} & & & 1 \\ & 1 & 1 & 2 \\ 1 & 2 & 3 & \\ 3 & 4 & & \end{array} \quad \longleftrightarrow \quad T = \begin{array}{cccc} 1 & 2 & 2 & 3 \\ 2 & 3 & & \\ 3 & 4 & & \\ 4 & & & \end{array}$$

$$U^* = \begin{array}{cccc} & & & 1 \\ & 1 & 1 & 2 \\ 2 & 2 & 3 & \\ 1 & 3 & & \end{array} \quad \longleftrightarrow \quad U = \begin{array}{cccc} 1 & 2 & 2 & 4 \\ 2 & 3 & 3 & \\ 3 & 4 & & \end{array}$$

The last example shows that the dual of a standard tableau need not be standard. Theorem 9.4 will show that such abnormalities cannot occur in our situation.

Our next preparatory result proves a close connection between right Capelli operators, dual pairs (T, T^*), right symmetrized bideterminants, and the dominance quasi-ordering.

Theorem 9.3 *Let A and B be two n-subsets of $\mathbb{N} \times \mathbb{N}$, and let α (resp. β) be the content of P_A (resp. P_B). Then the following holds.*

(a) $(S|P_A)R_{C(T)} = (S|T^*]$ *for every A-tableau S and every $T \in T^{A,\leq}(\beta)$.*

(b) $U \trianglelefteq V$ *iff* $U^* \trianglelefteq V^*$ *for all* $U, V \in T^{A,\leq}(\beta)$.

Proof.

(a). The proof is an easy exercise.

(b). For $U = (u_{ab})$ let $\rho_{ij}(U) := |\{p | u_{ip} = j\}|$. Then, by definition, $\rho_{ij}(U) = \rho_{ji}(U^*)$. Hence $r_{pq}(U) := |\{(i,j) | i \leq p, u_{ij} \leq q\}| = \sum_{i \leq p} \sum_{j \leq q} \rho_{ij}(U) = r_{qp}(U^*)$; consequently, $U \trianglelefteq V :\Leftrightarrow \forall p, q\ (r_{pq}(U) \leq r_{pq}(V)) \Leftrightarrow \forall p, q\ (r_{qp}(U^*) \leq r_{qp}(V^*)) \Leftrightarrow U^* \trianglelefteq V^*$. □

Theorem 9.4 *Let A be a skew diagram. If T is $\trianglelefteq$-maximal in* $\mathrm{supp}_r(X)$ *for some X in $S_A(R)$ then its A-dual T^* is a standard skew tableau.*

Proof. The right Capelli operator $R_{C(T)}$ does not annihilate X: $XR_{C(T)} \neq 0$. (To see this, write X as a linear combination of standard bideterminants and note that T is also $\Lsh^*$-maximal in $\mathrm{supp}_r(X)$. Our claim now follows from the first and second fundamental property of Capelli operators, see Theorems 3.3 and 3.6.)

Suppose T^* is not standard. By Theorem 9.1, $X = \sum_{S \in ST^A(1^n)} a_S (S|P_A)$ for suitable $a_S \in R$. Combining this with Theorem 9.3 (a) we get

$$0 \neq XR_{C(T)} = \sum_{S \in ST^A(1^n)} a_S (S|T^*]\ .$$

Since T^* is not a standard skew tableau, by Theorem 5.3 we can write $(S|T^*] = \sum_{W^*} b_{W^*} (S|W^*]$, where the sum is over all standard A-tableaux W^* satisfying $T^* \Lsh^* W^*$. (Note that the coefficients b_{W^*} do not depend on S but only on its injectivity.) Using Theorem 9.3 (b) and Theorem 3.6 and denoting the B-dual of W^* by W where $B :=$ shape(T), we get the following contradiction

$$\begin{aligned} 0 \neq X R_{C(T)} &= \textstyle\sum_S a_S (S|T^*] \\ &= \textstyle\sum_S a_S \sum_{T^* < W^*} b_{W^*} (S|W^*] \\ &= \textstyle\sum_S a_S \sum_{T<W} b_{W^*} (S|P_A) R_{C(W)} \\ &= X \cdot \left(\textstyle\sum_{T<W} b_{W^*} R_{C(W)} \right) = 0 \; . \end{aligned}$$

This proves the standardness of T^*. □

One consequence of our last result is that in the case of ordinary representation theory the multiplicity of the simple module $\mathcal{S}_B(R)$ in the skew module $\mathcal{S}_A(R)$ is $\leq |ST^B(\alpha)^* \cap ST^A(\beta)|$. We aim to prove that equality holds. This will be done by constructing appropriate elements in the skew module by means of the Laplace Duality Theorem. To this end we use those bijections $\varphi_T : B \to A$ which are encodings of dual pairs (T, T^*), where both T and T^* are standard. In order to describe the ith row of φ_T let $\{(p_1, q_1), \ldots, (p_r, q_r)\}$ denote the set of all $a = (p,q) \in A$ satisfying $T^*(a) = i$. Since T^* is standard we can arrange the (p_j, q_j) in such a way that both $p_1 \leq p_2 \leq \ldots \leq p_r$ and $q_1 > q_2 > \ldots > q_r$ hold. In our last example both T and T^* are standard. The corresponding φ_T and its inverse read as follows:

$$\varphi := \varphi_T = \begin{array}{llll} 14 & 23 & 22 & 31 \\ 24 & 32 & & \\ 33 & 41 & & \\ 42 & & & \end{array} = \begin{array}{llll} 1 & 2 & 2 & 3 \\ 2 & 3 & & \\ 3 & 4 & & \\ 4 & & & \end{array} \times \begin{array}{llll} 4 & 3 & 2 & 1 \\ 4 & 2 & & \\ 3 & 1 & & \\ 2 & & & \end{array} =: \varphi_1 \times \varphi_2$$

$$\psi := \varphi_{T^*} = \begin{array}{llll} & & & 11 \\ & 13 & 12 & 21 \\ 14 & 22 & 31 & \\ 32 & 41 & & \end{array} = \begin{array}{llll} & & & 1 \\ & 1 & 1 & 2 \\ 1 & 2 & 3 & \\ 3 & 4 & & \end{array} \times \begin{array}{llll} & & & 1 \\ & 3 & 2 & 1 \\ 4 & 2 & 1 & \\ 2 & 1 & & \end{array} =: \psi_1 \times \psi_2 \; .$$

A skew tableau C is called *co-standard* iff the entries in C are strictly decreasing from left to right in each row and weakly decreasing down each column. If $\varphi : X \to Y$ is a mapping between subsets X and Y of $\mathbb{N} \times \mathbb{N}$ then $\varphi(x) =: (\varphi_1(x), \varphi_2(x))$, $x \in X$, defines the projections $\varphi_i : X \to \mathbb{N}$. We will write $\varphi = \varphi_1 \times \varphi_2$. The following crucial definition goes back to James & Peel (1979) and Zelevinsky (1981a), (1981b), see also Clausen & Stötzer (1982), (1984).

Definition. A bijection $\varphi = \varphi_1 \times \varphi_2 : X \to Y$ between skew tableaux X and Y with inverse $\varphi^{-1} = \psi = \psi_1 \times \psi_2 : Y \to X$ is called a *picture* iff φ_1 and ψ_1 are standard, and φ_2 and ψ_2 are co-standard. $\mathcal{P}(X,Y)$ will denote the set of all pictures $\varphi : X \to Y$.

Note that $\varphi \in \mathcal{P}(X,Y)$ iff $\varphi^{-1} \in \mathcal{P}(Y,X)$. In our last example φ_T is a picture.

Theorem 9.5 *Let A and B be skew diagrams, $n = |A| = |B|$, and let α (resp. β) be the content of P_A (resp. P_B). Then $T \mapsto \varphi_T$ defines a bijection from $ST^B(\alpha) \cap ST^A(\beta)^*$ onto the set $\mathcal{P}(B,A)$ of all pictures from B to A.*

Proof. If T is in $ST^B(\alpha) \cap ST^A(\beta)^*$ then $\varphi_T =: \varphi = \varphi_1 \times \varphi_2$ is a bijection from B onto A and $\varphi_{T^*} =: \psi = \psi_1 \times \psi_2$ equals the inverse of φ_T. Moreover, $\varphi_1 = T$ and $\psi_1 = T^*$; hence both φ_1 and ψ_1 are standard skew tableaux. By construction, both φ_2 and ψ_2 are strictly decreasing from left to right in each row. Thus φ_T is a picture if and only if φ_2 and ψ_2 are weakly decreasing down each column. By way of contradiction assume that φ_2 does not have this property. Let the jth column be the first column of φ_2 violating the co-standardness of φ_2. Then $\varphi(p,j) = (a,b)$ and $\varphi(p+1,j) = (c,d)$ with $b < d$, for suitable p and j. Since φ_1 is standard we have $a < c$. Hence $\psi(a,b) = (p,j)$ and $\psi(c,d) = (p+1,j)$. Since A is a skew diagram, (a,d) belongs to A and the standardness of ψ_1 forces $\psi_1(a,d) = p$. Now ψ_2 is strictly decreasing in each row; thus $\psi(a,d) = (p,k)$ for some $k < j$. Consequently, as (p,k) and $(p+1,j)$ belong to the skew diagram B so does $(p+1,k)$. Hence the minimality of j yields the contradiction $d = \varphi_2(p,k) \geq \varphi_2(p+1,k) > \varphi_2(p+1,j) = d$. This proves that $T \mapsto \varphi_T$ maps $ST^B(\alpha) \cap ST^A(\beta)^*$ into $\mathcal{P}(B,A)$. Finally, a straightforward computation shows that $\varphi \mapsto \varphi_1$ is the inverse of $T \mapsto \varphi_T$. □

Now we are prepared to describe the relevant part of the right support of a skew module.

Theorem 9.6 *For a skew module $M = \mathcal{S}_A(R)$ the following sets are equal:*

$$\mathcal{G}_A := \{T | T \text{ is the } \trianglelefteq\text{-greatest element in } \operatorname{supp}_r(X) \text{ for some } X \in M\},$$

$$\mathcal{M}_A := \{T | T \text{ is a } \trianglelefteq\text{-maximal element in } \operatorname{supp}_r(X) \text{ for some } X \in M\},$$

$$\mathcal{P}_A := \{\psi_1 | \psi = \psi_1 \times \psi_2 : B \to A \text{ is a picture, } B \text{ a diagram}\}.$$

Proof. Trivially, $\mathcal{G}_A \subseteq \mathcal{M}_A$. By Theorem 9.4 and Theorem 9.5, $\mathcal{M}_A \subseteq \mathcal{P}_A$. We finally prove that $\mathcal{P}_A \subseteq \mathcal{G}_A$: Let B be a diagram and let $\varphi : A \to B$ denote the inverse of the picture $\psi : B \to A$. Then $T' := \psi_1$ is a standard B-tableau. Let S' be any fixed standard B-tableau of content (1^n), $n = |A| = |B|$. If

$S := S' \circ \varphi$, $T := T' \circ \varphi$ then, by Laplace Duality (see Theorem 4.1 and its notation), $(S;\varphi|T) = (S'|T';\psi)$. Since $T = T' \circ \varphi = \psi_1 \circ \varphi = P_A$, $(S;\varphi|T)$ is a signed sum of bideterminants of the form $(S \circ \sigma|P_A)$; thus $(S;\varphi|T) \in \mathcal{S}_A(R)$. In order to finish the proof it suffices to show that

$$(S'|T';\psi) = (S'|T') + \sum \eta_{VW}(V|W) \ ,$$

where the sum is over all standard bitableaux (V, W) such that W is strictly dominated by T', $W \lhd T'$. (Since (S', T') is a standard bitableau this then shows that T' is the $\trianglelefteq$-greatest element in the right support of $(S'|T';\psi)$.) According to section 4, $(S'|T';\psi) = \sum_\tau \operatorname{sgn}(\tau)(S'|T' \circ \tau)$, where τ runs through all φ-shuffles. Since the identity of B is a φ-shuffle, one of the above summands equals $(S'|T')$. We prove that for all φ-shuffles $\tau \neq \mathrm{id}_B$, the expansion of $(S'|T' \circ \tau)$ as a linear combination of standard bideterminants only involves standard bitableaux (V, W) satisfying $W \lhd T'$. W.l.o.g. we can assume that $(S'|T' \circ \tau) \neq 0$, i.e. $T' \circ \tau$ is column-injective. Since, by Theorem 3.7, W is dominated by the standardization $(T' \circ \tau)^{st}$ of $T' \circ \tau$, it is enough to show that $(T' \circ \tau)^{st}$ is strictly dominated by T'. Let A^j denote the jth column of A. Then $C_j := \varphi[A^j]$ is a skew diagram contained in B. In addition, since φ_1 is standard, each row of C_j has length at most 1. Furthermore $T' \downarrow C_j$ is a standard skew tableau whose content γ equals that of $\varphi_1 \downarrow A^j$. Thus if τ runs through all φ-shuffles, then $T' \circ \tau \downarrow C_j$ runs through all standard skew tableaux of shape C_j and content γ. Since $T' \downarrow C_j$ is an order monomorphism from $(C_j, \leq_t)$ into $(\mathbb{N}, \leq)$, $T' \downarrow C_j$ is strictly column-dominated by $T' \circ \tau \downarrow C_j$, for every φ-shuffle $\tau \neq 1$. Hence, if $\tau \neq 1$ is a φ-shuffle and if $T' \circ \tau \circ \pi$ is a column-strict B-tableau, where $\pi \in \mathcal{V}(B)$, then $T' \lhd_c T' \circ \tau \circ \pi \trianglelefteq_c (T' \circ \tau)^{st} =: Z$. By section 3 this implies $T' \rhd Z$, and the proof of Theorem 9.6 is complete. ◻

The last theorem and its proof enable us to describe for every skew module $\mathcal{S}_A(R)$ an R-basis that is adapted to a Specht series. If $\psi : B \to A$ is a picture, B a diagram, let B_ψ denote the set of all $(S'|\psi_1;\psi)$, where S' runs through all standard B-tableaux of content (1^n), $n = |A| = |B|$.

Theorem 9.7 *Let A be a skew diagram. Then the union of all B_ψ, where ψ runs through all pictures in*

$$\operatorname{Pic}(A) := \bigcup_{B \text{ a diagram}} \mathcal{P}(B, A) \ ,$$

is an R-basis of the skew module $\mathcal{S}_A(R)$. Under a suitable ordering, this basis is adapted to a Specht series of the skew module $\mathcal{S}_A(R)$.

Proof. Let $\mathcal{B}_A$ denote the union of all B_ψ. The last theorem combined with the linear independence of the standard bideterminants implies the linear independence of $\mathcal{B}_A$. Next we show that every element X in the skew module can be written as a linear combination of the elements in $\mathcal{B}_A$. Let $X = \sum \xi_{UV}(U|V)$ be the expansion of X as a linear combination of standard bideterminants. Now let $\leq$ be a fixed linearization of the dominance ordering $\trianglelefteq$ on the set of all standard tableaux whose content equals the content of P_A. Furthermore let T' be the $\leq$-greatest element in $\mathrm{supp}_r(X)$. By Theorems 9.4, 9.5, and 9.6 there is a unique picture ψ whose first projection equals T', i.e. $\psi_1 = T'$. If $I := \{S' | \xi_{S'T'} \neq 0\}$ then by the proof of Theorem 9.6

$$X' := X - \sum_{S' \in I} \xi_{S'T'}(S'|T';\psi) \in \mathcal{S}_A(R) .$$

If $X' = 0$ we are done. Otherwise we replace X by X' and continue as above. Since the $\leq$-greatest element in $\mathrm{supp}_r(X')$ is strictly smaller than T' and since the right support of the skew module is finite, this process must terminate.

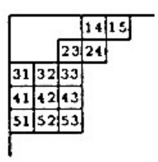
Finally we show that under a suitable ordering the basis consisting of the union of all B_ψ is adapted to a Specht series of the skew module. Let $\psi^1, \ldots, \psi^r$ be all elements in $\mathrm{Pic}(A)$, and let T_j denote the first projection of ψ^j. W.l.o.g. we can assume that $T_1 < T_2 < \ldots < T_r$. Let M_j be the R-linear hull of all elements in $B_{\psi^1} \cup \ldots B_{\psi^j}$. By Corollary 3.8 and Theorem 9.6, $0 =: M_0 \subset M_1 \ldots \subset M_r = \mathcal{S}_A(R)$ is a chain of RS_n-submodules of $\mathcal{S}_A(R)$, and the right Capelli operator $R_{D(T_j)}$ maps M_j onto the Specht module $\mathcal{S}_{\lambda^j}(R)$ if $\lambda^j := \mathrm{shape}(T_j)$; furthermore, M_{j-1} is the kernel of $R_{D(T_j)} \downarrow M_j$. Hence $M_0 \subset M_1 \ldots \subset M_r$ is a Specht series of the skew module $\mathcal{S}_A(R)$. □

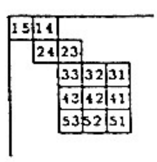
The next section presents an algorithm which constructs for a given skew diagram A all pictures in $\mathrm{Pic}(A)$.

10 Pictures

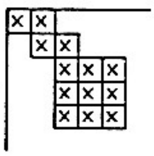
Let A be a skew diagram. In order to generate $\mathrm{Pic}(A)$ we associate to A a tree whose leaves irredundantly describe $\mathrm{Pic}(A)$. Our algorithm is quite simple. It essentially consists of a description of the root T_A and the *hook deformation rule* by which all descendants of a node in that tree are constructed.

To get the root T_A we simply reverse the order of the columns in the identity id_A of A:

$\mathrm{id}_A =$ 

$T_A =$ 

To describe the hook deformation rule, we need some preparations. For $(r,s) \in \mathbb{N} \times \mathbb{N}$ the set $H(r,s) := \{(r+i, s+j) | i,j \geq 0\}$ is called the *hook* w.r.t. (r,s). Let T be a node in our tree corresponding to A. As a matter of fact, T is a bijection, mapping the domain, denoted by $|T|$, onto A, thus $T : |T| \to A$. In the example above

$| T_A | =$ 

In order to construct all descendants of T we first have to compute the greatest diagram D contained in $|T|$. If $D = |T|$ then T is a leaf. Otherwise let (r,s) be the leftmost element in the topmost row of $|T| \backslash D$. If $(r_1, s_1), \ldots, (r_d, s_d)$ are all elements in $\mathbb{N} \times \mathbb{N}$ satisfying for all $1 \leq k \leq d$

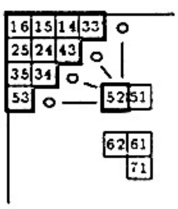
$r_k \leq r, s_k \leq s,$

$(r_k, s_k) \notin D$, and

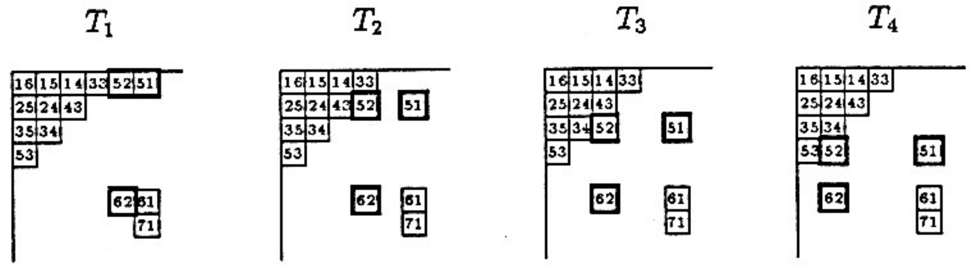
$D \cup \{(r_k, s_k)\}$ is a diagram,

then T has d descendants $T_1, \ldots, T_d$. For $1 \leq k \leq d$ the descendant T_k results from T by translating the portion in T corresponding to the hook $H(r,s)$ into the hook $H(r_k, s_k)$. (It can be shown that $H(r_k, s_k)$ and $|T|$ are disjoint sets.) T and T_k are equal outside $H(r,s) \cup H(r_k, s_k)$. The hook deformation $T \downarrow (H(r,s) \cap |T|) \to T_k \downarrow (H(r_k, s_k) \cap |T_k|)$ is a bijection that is defined as follows: if $(r, s+j) \in |T|$ and $j > 0$ then $(r_k, s+j) \in |T_k|$ and $T_k(r_k, s+j) := T(r, s+j)$. Analogously, if $(r+i, s) \in |T|$ and $i > 0$ then $(r+i, s_k) \in |T_k|$ and $T_k(r+i, s_k) := T(r+i, s)$. Finally, $T_k(r_k, s_k) := T(r,s)$.

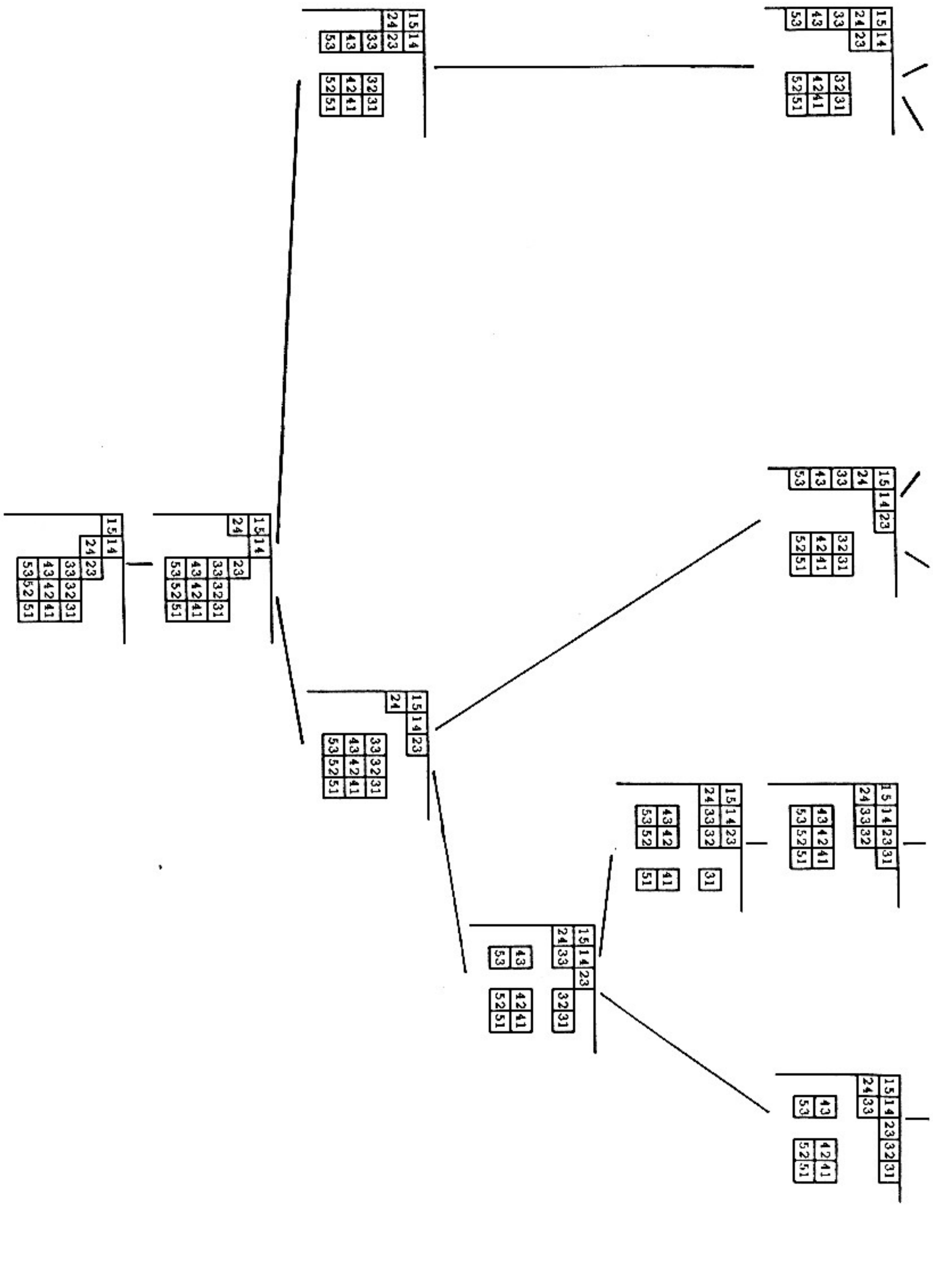
For a detailed version of this algorithm and its verification the reader is referred to Clausen & Stötzer (1982),(1984). We illustrate this algorithm by an example. If

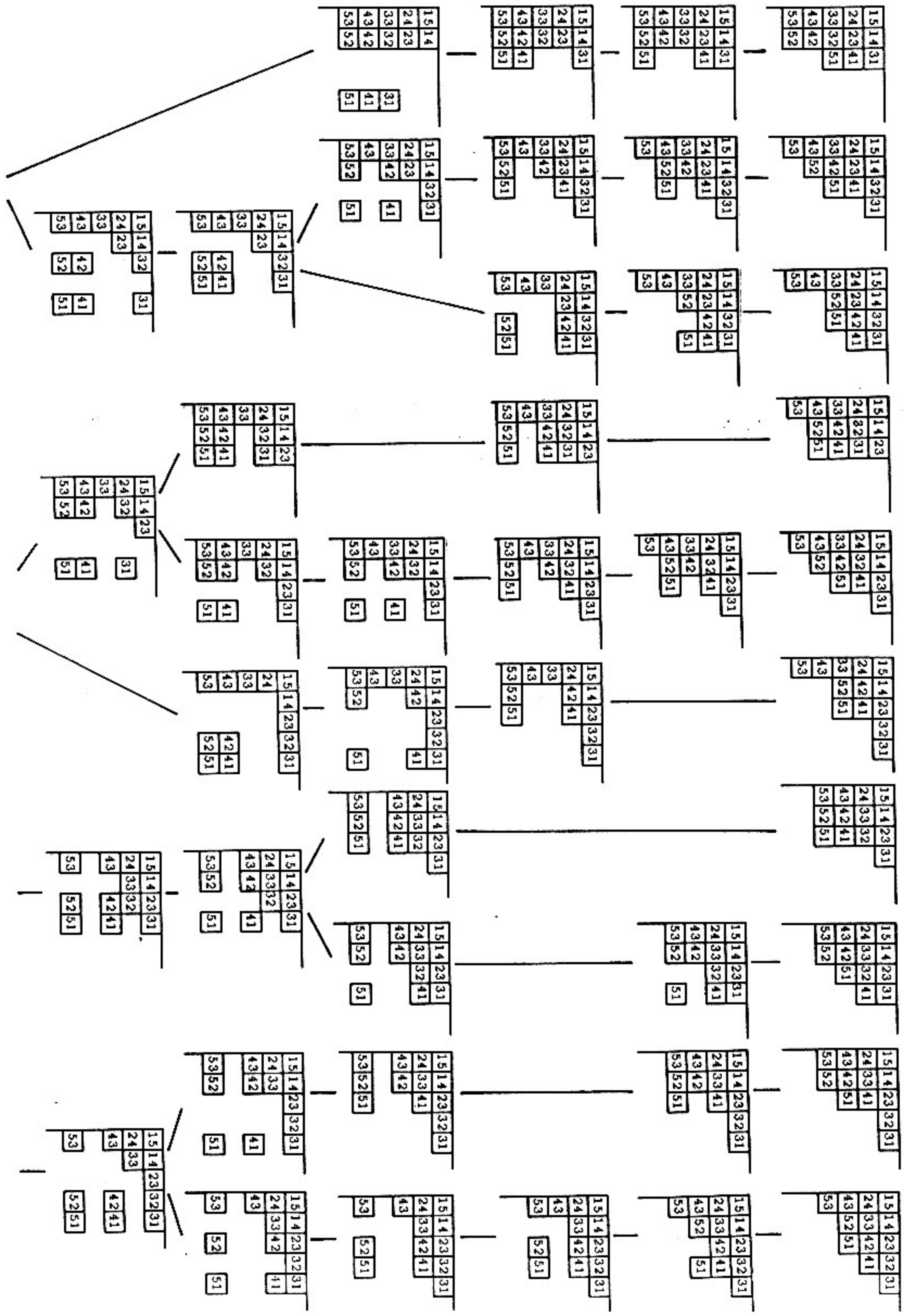
$T =$ 

then $(r,s) = (4,5)$, $T(4,5) = (5,2)$ and $(r_1,s_1) = (1,5), (r_2,s_2) = (2,4)$, $(r_3,s_3) = (3,3), (r_4,s_4) = (4,2)$. Hence T has four descendants T_1, T_2, T_3, T_4 which read as follows



Our final example shows such a tree. Recall that the leaves of this tree describe Pic(A).





References

Abeasis, S., Del Fra, A. (1980). Young diagrams and ideals of Pfaffians, *Advances in Math.*, **35**, 158–178.

Abhyankar, S. (1988). Enumerative Combinatorics of Young Tableaux, New York: Marcel Dekker.

Barnabei, M., Brini, A. (1987). Symmetrized skew determinants, *Communications in Algebra*, **15**(7), 1455–1468.

Carter, R. W., Lusztig, G. (1974). On the modular representations of the general linear and symmetric groups, *Math. Z.*, **136**, 193–242.

Clausen, M. (1979). Letter place algebras and a characteristic-free approach to the representation theory of the general linear and symmetric groups, I, *Advances in Math.*, **33**, 161–191.

Clausen, M. (1980a). Letter place algebras and a characteristic-free approach to the representation theory of the general linear and symmetric groups, II, *Advances in Math.*, **38**, 152–177.

Clausen, M. (1980b). Letter-Place-Algebren und ein charakteristik-freier Zugang zur Darstellungstheorie symmetrischer und voller linearer Gruppen, *Bayreuther Mathematische Schriften*, **4**, 151 S.

Clausen, M. (1984). Dominance orders, Capelli operators, and straightening of bideterminants, *Europ. J. Combinatorics*, **5**, 207–222.

Clausen, M., Grabmeier J.(1990). On a class of cyclic S_n-modules, in preparation.

Clausen, M., Stötzer, F.(1982). Pictures and skew (reverse) plane partitions, *Springer Lecture Notes in Mathematics*, **969**, 100–114.

Clausen, M., Stötzer, F. (1984). Pictures und Standardtableaux—Grundlagen und Algorithmen, *Bayreuther Mathematische Schriften*, **16**, 1–122.

De Concini, C., Procesi, C. (1976). A characteristic free approach to invariant theory, *Advances in Math.*, **21**, 330–354.

De Concini, C. (1979). Symplectic standard tableaux, *Advances in Math.*, **34**, 1–27.

De Concini, C., Eisenbud, D., Procesi, C. (1980). Young diagrams and determinantal varieties, *Inventiones Math.*, **56**, 129–165.

De Concini, C., Strickland, E. (1979). Traceless tensors and the symmetric group, *J. Algebra*, **61**, 112–128.

De Concini, C., Strickland, E. (1981). On the variety of complexes, *Advances in Math.*, **41**, 57–77.

Désarménien, J. (1980). An algorithm for the Rota straightening formula, *Discrete Math.*, **30**, 51–68.

Désarménien, J., Kung, J. P. S., Rota, G.-C. (1978). Invariant theory, Young bitableaux and combinatorics, *Advances in Math.*, **27**, 63–92.

Dirl, R., Kasperkovitz, P. (1977). Gruppentheorie, Anwendungen in der Atom- und Festkörperphysik, Braunschweig, FRG: Vieweg.

Doubilet, P., Rota, C.-G. (1976). Skew-symmetric invariant theory, *Advances in Math.*, **21**, 196–201.

Doubilet, P., Rota, C.-G., Stein, J. (1974). On the foundations of combinatorial theory: IX. Combinatorial methods in invariant theory, *Stud. Appl. Math.*, **53**, 185–216.

Golembiowski, A. (1987). Zur Berechnung modular irreduzibler Matrixdarstellungen symmetrischer Gruppen mit Hilfe eines Verfahrens von M. Clausen, *Bayreuther Mathematische Schriften*, **25**, 135–222.

Green, J. A. (1980). Polynomial representations of GL_n, *Springer Lecture Notes in Mathematics*, **830**.

James, G.D. (1977). A characteristic-free approach to the representation theory of S_n, *J. Algebra* **46**, 430–450.

James, G.D., Kerber, A. (1981). The representation theory of the symmetric group, Encyclopedia of Mathematics and its Applications **16**, Reading, MA: Addison Wesley.

James, G.D., Peel, M.H. (1979). Specht series for skew representations of symmetric groups, *J. Algebra*, **56**, 343–364.

Knuth, D. E. (1970). Permutations, matrices and generalized Young tableaux, *Pacific J. Math.*, **34**, 709–727.

Kramer, P., John, G., Schenzle, D. (1981). Group theory and the interaction of composite nucleon systems, Braunschweig, FRG: Viehweg.

Kung, J. P. S., Rota, G.-C. (1981). On the differential invariants of a linear ordinary differential equation, *Proc. Roy. Soc. Edinburgh*, **89A**, 111–123.

Louck, J. S., Biedenharn, L. C. (1973). On the structure of the canonical tensor operators in the unitary groups. III. Further developments of the boson polynomials and their applications, *J. Math. Phys.*, **14**, 1336–1357.

Macdonald, I.G. (1979). Symmetric Functions and Hall Polynomials, Oxford: University Press.

Mead, D. G. (1972). Determinantal ideals, identities and the Wronskian, *Pacific J. Math.*, **42**, 167–175.

Pittaluga, M., Strickland, E. (1988). On computers and modular representations of $SL_n(K)$, *Lecture Notes in Computer Science*, **307**, 120–129, Heidelberg, FRG: Springer.

Pommering, K. (1981). Invarianten unipotenter Gruppen, *Math. Z.*, **176**, 359–374.

Rota, G.-C., Désarménien, J. (1977). Théorie combinatoire des invariants classiques, Series de Mathématique Pures et Appliquées, IRMA, Strasbourg.

Schensted, C. (1961). Longest increasing and decreasing subsequences, *Canad. J. Math.*, **13**, 179–191.

Schützenberger, M. P. (1963). Quelques remarques sur une construction de Schensted, *Math. Scand.*, **12**, 117–128.

Stanley, R.P. (1971). Theory and Application of Plane Partitions, Part 1, *Studies in Applied Mathematics*, **50**, 167–188.

Stanley, R.P. (1977). Some Combinatorial Aspects of Schubert Calculus, *Lecture Notes in Mathematics*, **579**, Heidelberg, FRG: Springer.

Stein, J. (1980). Hopf Algebra Structures on Tensor Products of **Z**-Modules, Thesis, Harvard University.

Strickland, E. (1980). The symplectic group and determinants, *J. Algebra*, **66**, 511–533.

Zelevinsky, A.V. (1981a). A generalization of the Littlewood-Richardson rule and the Robinson-Schensted-Knuth correspondence, *J. Algebra*, **69**, 82–94.

Zelevinsky, A.V. (1981b). Representations of finite classical groups—a Hopf algebra approach, Lecture Notes in Mathematics, **869**, Heidelberg, FRG: Springer.