Lower Bound on Testing Membership to a Polyhedron by Algebraic Decision Trees

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Abstract

We introduce a new method of proving lower bounds on the depth of algebraic d-degree decision trees and apply it to prove a lower bound $\Omega(\log N)$ for testing membership to an n-dimensional convex polyhedron having N faces of all dimensions, provided that $N > (nd)^{\Omega(n)}$. This bound apparently does not follow from the methods developed by M. Ben-Or, A. Björner, L. Lovasz, and A. Yao [B. 83], [BLY 93], [Y 94] because topological invariants used in these methods become trivial for convex polyhedra.

Introduction

A problem of testing membership to a semialgebraic set Σ was considered by many authors (see, e.g., [B 83], [B 92], [BKL 92], [BL 92], [BLY 92], [MH 85], [GKV 94], [Y 92], [Y 93], [YR 80] and the references there). We consider a problem of testing membership to a convex polyhedron P in n-dimensional space $\mathbf{R}^n$. Let P have N faces of all the dimensions. In [MH 85] it was shown, in particular, that for this problem $O(\log N)n^{O(1)}$ upper bound is valid for the depth of linear decision trees, in [YR 80] a lower bound $\Omega(\log N)$ was obtained. A similar question was open for algebraic decision trees. In [GKV 94] we proved a lower bound $\Omega(\log N)$ for the depth of algebraic decision trees testing membership to P, provided that $N > (dn)^{\Omega(n^2)}$. In the present paper we weaken the latter assumption to $N > (dn)^{\Omega(n)}$. In this new form the bound looks plausible to be applicable to polyhedra given by $2^{O(n)}$ linear constraints (like in "knapsack" problem), thus having $2^{O(n^2)}$ faces. In the present paper we apply the obtained lower bound to a concrete class of polyhedra given by $\Omega(n^2)$ linear constraints and with $n^{\Omega(n)}$ faces.

In [GV 94] the lower bound $\Omega(\sqrt{\log N})$ was proved for the Pfaffian computation tree model. This model uses at gates *Pfaffian functions*, the latter include all major elementary transcendental and algebraic functions.

Several topological methods were introduced for obtaining lower bounds for the complexity of testing membership to Σ by linear decision trees, algebraic decision trees, algebraic computation trees (the definitions one can find in, e.g., [B 83]).

In [B 83] a lower bound $\Omega(\log C)$ was proved for the most powerful among the considered in this area computational models, namely algebraic computation trees, where C is the number of connected components of Σ or of the complement of Σ. After that, in [BLY 92], a lower bound $\Omega(\log \chi)$ for linear decision trees was proved, where χ is Euler characteristic of Σ, in [Y 92] this lower bound was extended to algebraic computation trees. A stronger lower bound $\Omega(\log B)$ was proved later in [BL 92], [B 92] for linear decision trees, where B is the sum of Betti numbers of Σ (obviously, $C, \chi \leq B$). In [Y 94] the latter lower bound was extended to the algebraic decision trees.

Unfortunately, all the mentioned topological tools fail when Σ is a convex polyhedron, because $B = 1$ in this situation. The same is true for the method developed in [BLY 92] for linear decision trees, based on the minimal number of convex polyhedra onto which Σ can be partitioned.

To handle the case of a convex polyhedron, we introduce in Sections 1, 3 another approach which differs drastically from [GKV 94]. Let W be a semialgebraic set accepted by a branch of an algebraic decision tree. In Section 3 we make an "infinitesimal perturbation" of W which transforms this set into a smooth hypersurface. Then we describe the semialgebraic subset of all the points of the hypersurface in which all its principal curvatures are "infinitely large" (the set $\mathcal{K}_0$ in Section 3). We also construct a more general set $\mathcal{K}_i$ (for each $0 \leq i \leq n-1$) of the points with infinitely large curvatures in the intersections with the shifts of a fixed $(n-i)$-dimensional plane. Section 1 provides a short system of inequalities for determining $\mathcal{K}_i$. It is done by developing an explicit symbolic calculus for principal curvatures.

In Section 2 we introduce some necessary notions concerning infinitesimals and apply them to define the "standard part" $K_i = \mathrm{st}(\mathcal{K}_i) \subset \mathbf{R}^n$. We show (Corollary to Lemma 5 in Section 3) that to obtain the required bound for the number of i-faces P_i of P such that $\dim(P_i \cap W) = i$ it is sufficient to estimate the number of faces P_i with $\dim(P_i \cap K_i) = i$. In Section 4 we reduce the latter bound to an estimate of the number of local maxima of a generic linear function L on $\mathcal{K}_i$ with the help of a Whitney stratification of K_i. To estimate these local maxima we introduce in Section 5 another infinitesimal perturbation of $\mathcal{K}_i$ and obtain a new smooth hypersurface. At this point a difficulty arises due to the fact that $\mathcal{K}_i$ (and therefore, the related smooth hypersurface) are defined by systems of inequalities involving algebraic functions, rather than polynomials, because in the expressions for curvatures (in Section 1) square roots of polynomials appear. We represent the set of local maxima of L on the smooth hypersurface by a formula of the first-order theory of real closed fields with merely existential quantifiers and quantifier-free part Φ. We estimate in Section 5 (invoking [Mi 64] in a usual way) the number of the connected components of the semialgebraic set defined by Φ.

In Section 6 we describe a particular class of polyhedra (dual to cyclic polyhedra [MS 71]) having large numbers of faces, for which Theorem 1 provides a nontrivial lower bound.

Now let us formulate precisely the main result. We consider algebraic decision trees of a fixed degree d (see, e.g., [B 83], [Y 93]). Suppose that such a tree T, of the depth k, tests a membership to a convex polyhedron $P \subset \mathbf{R}^n$. Denote by N the number of faces of P of all dimensions from zero to $n-1$. In this paper we agree that a face is "open", i.e., does not contain faces of smaller dimensions.

Theorem 1.

$$k \geq \Omega(\log N),$$

provided that $N \geq (dn)^{cn}$ *for a suitable* $c > 0$.

Let us fix a branch of T which returns "yes". Denote by $f_i \in \mathbf{R}[X_1, \dots, X_n]$, $1 \leq i \leq k$ the polynomials of degrees $\deg(f_i) \leq d$, attached to the vertices of T along the fixed branch. Without loss of generality, we can assume that the corresponding signs of polynomials along the branch are

$$f_1 = \cdots = f_{k_1} = 0,\ f_{k_1+1} > 0, \dots, f_k > 0.$$

Then the (accepted) semialgebraic set

$$W = \{f_1 = \cdots = f_{k_1} = 0,\ f_{k_1+1} > 0, \dots, f_k > 0\}$$

lies in P.

Our main technical tool is the following theorem.

Theorem 2. *The number of faces P' of P such that* $\dim(P') = \dim(P' \cap W)$ *is bounded from above by* $(knd)^{O(n)}$.

Let us deduce Theorem 1 from Theorem 2.

For each face P' of P there exists at least one branch of the tree T with the output "yes" and having an accepted set $W_1 \subset \mathbf{R}^n$ such that

$$\dim(W_1 \cap P') = \dim(P').$$

Since there are at most 3^k different branches of T, the inequality

$$N < 3^k (knd)^{O(n)}$$

follows from Theorem 2. This inequality and the assumption $N > (dn)^{cn}$ (for a suitable c) imply $k \geq \Omega(\log N)$, which proves Theorem 1.

Note that in the case $k_1 = 0$ for an open set W and each face P' of P we have $P' \cap W = \emptyset$. Thus in what follows we can suppose that $k_1 \geq 1$.

1. Computer algebra for curvatures

Let a polynomial $F \in \mathbf{R}[X_1, \dots, X_n]$ with $\deg(F) < d$. Assume that at a point $x \in \{F = 0\} \subset \mathbf{R}^n$ the gradient $\mathrm{grad}_x(F) = \left(\frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_n}\right)(x) \neq 0$. Then, according to the implicit function theorem, the real algebraic variety $\{F = 0\} \subset \mathbf{R}^n$ is a smooth hypersurface in a neighbourhood of x.

Fix a point $x \in \{F = 0\}$. Consider a linear transformation $X \longrightarrow AX + x$, where A is an arbitrary orthogonal matrix such that

$$u_1 = Ae_1 + x = \frac{\mathrm{grad}_x(F)}{\|\mathrm{grad}_x(F)\|}$$

is the normalized gradient and $e_1, \dots, e_n$ is the coordinate basis at the origin. Then the linear hull of vectors $u_j = Ae_j + x$, $2 \leq j \leq n$ is the tangent space T_x to $\{F = 0\}$ at x.

Denote by $U_1, \dots, U_n$ the coordinate variables in the basis $u_1, \dots, u_n$. By the implicit function theorem, there exists a smooth function $H_x(U_2, \dots, U_n)$ defined in a neighbourhood of x on T_x such that $\{F = 0\} = \{U_1 = H_x(U_2, \dots, U_n)\}$ in this neighbourhood.

Let $u_1 = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_n)$ with $\tilde{\alpha}_{i_0} \neq 0$. Take any permutation π_{i_0} of $\{1, \dots, n\}$ such that $\pi_{i_0}(1) = i_0$. Denote $(\alpha_1, \dots, \alpha_n) = (\tilde{\alpha}_{\pi_{i_0}(1)}, \dots, \tilde{\alpha}_{\pi_{i_0}(n)})$ (thus $\alpha_1 \neq 0$) and $\beta_i = \sqrt{\alpha_1^2 + \cdots + \alpha_i^2}$, $1 \leq i \leq n$. Obviously $\beta_i > 0$ and $\beta_n = 1$.

As A one can take the following product of $(n-1)$ orthogonal matrices:

$$\prod_{0 \leq k \leq n-2} \begin{pmatrix} \frac{\beta_{n-k-1}}{\beta_{n-k}} & 0 & \cdots & 0 & \frac{\alpha_{n-k}}{\beta_{n-k}} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ -\frac{\alpha_{n-k}}{\beta_{n-k}} & 0 & \cdots & 0 & \frac{\beta_{n-k-1}}{\beta_{n-k}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

(in kth matrix of this product the element $\frac{\beta_{n-k-1}}{\beta_{n-k}}$ occurs at the positions $(1, 1)$ and $(n - k, n - k)$).

Denote $F_x(U_1, \dots, U_n) = F(A^T(U_1, \dots, U_n) + x)$. Differentiating this function twice and taking into the account that $F_x(H_x(U_2, \dots, U_n), U_2, \dots, U_n) = 0$ in a neighbourhood of x in T_x we get

$$\frac{\partial^2 F_x}{\partial U_1 \partial U_j} \frac{\partial H_x}{\partial U_i} + \frac{\partial F_x}{\partial U_1} \frac{\partial^2 H_x}{\partial U_i \partial U_j} + \frac{\partial^2 F_x}{\partial U_i \partial U_j} = 0 \tag{1}$$

for $2 \le i,\ j \le n$.

Since

$$\frac{\partial H_x}{\partial U_i}\Big|_{(U_2,\dots,U_n)=0} = 0 \text{ and } \frac{\partial F_x}{\partial U_1}\Big|_{(U_1,\dots,U_n)=0} = \|\mathrm{grad}_x(F)\| \neq 0,$$

evaluating the equality (1) at x (i.e., substituting $(U_1,\dots,U_n)=0$) we obtain (cf. [Mi 64]):

$$\Big(\frac{\partial^2 H_x}{\partial U_i \partial U_j}\Big)\Big|_{(U_2,\dots,U_n)=0} = (\|\mathrm{grad}_x(F)\|)^{-1}\Big(\frac{\partial^2 F_x}{\partial U_i \partial U_j}\Big)\Big|_{(U_1,\dots,U_n)=0}. \tag{2}$$

Introduce the symmetric $(n-1)\times(n-1)$-matrix (the matrix of Weingarten map [Th 77], Ch.9)

$$\mathcal{H}_x = \Big(\frac{\partial^2 H_x}{\partial U_i \partial U_j}\Big)\Big|_{(U_2,\dots,U_n)=0}.$$

Its eigenvalues $\lambda_2,\dots,\lambda_n$ belong to $\mathbf{R}$ and are called the *principal curvatures* of the hypersurface $\{F=0\}$ at x [Th 77], Ch.12.

Now we describe symbolically the set of all points x with all principal curvatures greater than some parameter κ.

Denote by $\chi(Z)$ the characteristic polynomial of the matrix $\mathcal{H}_x$. The roots of χ are exactly $\lambda_2,\dots,\lambda_n$. Due to Sturm theorem, every $\lambda_2,\dots,\lambda_n$ is greater than κ if and only if $\chi_l(\kappa)\chi_{l+1}(\kappa) < 0,\ 0 \le l \le n-2$, where $\chi_0 = \chi,\ \chi_1 = \chi_0'$ and $\chi_2,\dots,\chi_{n-1}$ is the polynomial remainder sequence of $\chi_0,\ \chi_1$ [Lo 82]. Obviously $\deg_Z(\chi_l) = n-l-1$.

Observe that every element of the matrix A can be represented as a fraction γ_1/γ_2 where

$$\gamma_2 = \beta_1^{\nu_1}\cdots\beta_{n-1}^{\nu_{n-1}}\|\mathrm{grad}_x(F)\|^{\nu}$$

and $\gamma_1 = \Gamma(\beta_1,\dots,\beta_{n-1},X_1,\dots,X_n)$ is a polynomial in

$$\beta_1(X_1,\dots,X_n),\dots,\beta_{n-1}(X_1,\dots,X_n),X_1,\dots,X_n$$

with $\Gamma \in \mathbf{R}[Z_1,\dots,Z_{n-1},X_1,\dots,X_n]$. Moreover, $\nu_1+\cdots+\nu_{n-1}+\nu \le 2(n-1)$ and $\deg(\Gamma) \le d(n-1)$. Hence all elements of A are algebraic functions in $X_1,\dots,X_n$ of quadratic-irrational type. By the degree of such quadratic-irrational function we mean $\max\{\deg(\Gamma),\ \nu_1+\cdots+\nu_{n-1}+\nu\}$. In what follows we deal with algebraic functions in $X_1,\dots,X_n$ of the similar type.

Formula (2) and Habicht's theorem [Lo 82] imply that $\deg(\chi_l) \le (nd)^{O(1)}$.

We summarize a description of the set of all points with large principal curvatures in the following lemma.

Lemma 1. *Fix $1 \le i_0 \le n$. The set of all points $x \in \{F=0\}$ such that $\mathrm{grad}_x(F) = (\hat\alpha_1,\dots,\hat\alpha_n)$ has $\hat\alpha_{i_0} \neq 0$ and all principal curvatures of the hypersuface $\{F=0\}$ at x are greater than κ can be represented as $\{F=0,\ g_1>0,\dots,g_n>0\}$. Here $g_1 = \hat\alpha_{i_0}^2, g_2,\dots,g_n$ are polynomials in κ of degrees at most $2n$ with coefficients being quadratic-irrational algebraic functions (see above) of degrees less than $(nd)^{O(1)}$.*

Remark. Observe that a set given by a system of inequalities involving real algebraic functions is semialgebraic. Hence the set introduced in Lemma 1 is semialgebraic.

2. Calculis with infinitesimals

The following definitions concerning infinitesimals follow [GV 88].

Let $\mathbf{F}$ be an arbitrary real closed field (see, e.g., [L 65]) and an element ε be infinitesimal relative to elements of $\mathbf{F}$. The latter means that for any positive element $a \in \mathbf{F}$ inequalities $0 < \varepsilon < a$ are valid in the ordered field $\mathbf{F}(\varepsilon)$. Obviously, the element ε is transcendental over $\mathbf{F}$. For an ordered field $\mathbf{F}'$ we denote by $\tilde{\mathbf{F}}'$ its (unique up to isomorphism) real closure, preserving the order on $\mathbf{F}'$ [L 65].

Let us remind some other well-known statements concerning real closed fields. A Puiseux (formal power-fractional) series over $\mathbf{F}$ is series of the kind

$$b = \sum_{i \geq 0} a_i \varepsilon^{\nu_i/\mu},$$

where $0 \neq a_i \in \mathbf{F}$ for all $i \geq 0$, integers $\nu_0 < \nu_1 < \ldots$ increase and the natural number $\mu \geq 1$. The field $\mathbf{F}((\varepsilon^{1/\infty}))$ consisting of all Puiseux series (appended by zero) is real closed, hence $\mathbf{F}((\varepsilon^{1/\infty})) \supset \widetilde{\mathbf{F}(\varepsilon)} \supset \mathbf{F}(\varepsilon)$. Besides the field $\mathbf{F}[\sqrt{-1}]((\varepsilon^{1/\infty}))$ is algebraically closed.

If $\nu_0 < 0$, then the element $b \in \mathbf{F}((\varepsilon^{1/\infty}))$ is infinitely large. If $\nu_0 > 0$, then b is infinitesimal relative to elements of the field $\mathbf{F}$. A vector $(b_1, \ldots, b_n) \in \left(\mathbf{F}((\varepsilon^{1/\infty}))\right)^n$ is called $\mathbf{F}$-finite if each coordinate b_i, $1 \leq i \leq n$ is not infinitely large relative to elements of $\mathbf{F}$.

For any $\mathbf{F}$-finite element $b \in \mathbf{F}((\varepsilon^{1/\infty}))$ its *standard part* $\mathrm{st}(b)$ is definable, namely $\mathrm{st}(b) = a_0$ in the case $\nu_0 = 0$ and $\mathrm{st}(b) = 0$ if $\nu_0 > 0$. For any $\mathbf{F}$-finite vector $(b_1, \ldots, b_n) \in \left(\mathbf{F}((\varepsilon^{1/\infty}))\right)^n$ its standard part is defined by the equality

$$\mathrm{st}(b_1, \ldots, b_n) = (\mathrm{st}(b_1), \ldots, \mathrm{st}(b_n)).$$

For a set $\mathcal{W} \subset \left(\mathbf{F}((\varepsilon^{1/\infty}))\right)^n$ we define

$$\mathrm{st}(\mathcal{W}) = \{\mathrm{st}(w) : \ w \in \mathcal{W} \text{ and } w \text{ is } \mathbf{F}\text{-finite}\}.$$

The following "transfer principle" is true [T 51]. If $\mathbf{F}'$, $\mathbf{F}''$ are real closed fields with $\mathbf{F}' \subset \mathbf{F}''$ and $\mathcal{P}$ is a closed (without free variables) formula of the first order theory of the field $\mathbf{F}'$, then $\mathcal{P}$ is true over $\mathbf{F}'$ if and only if $\mathcal{P}$ is true over $\mathbf{F}''$.

In the sequel we consider infinitesimals $\varepsilon_1, \varepsilon_2, \ldots$ such that ε_{i+1} is infinitesimal relative to the real closure $\mathbf{R}_i$ of the field $\mathbf{R}(\varepsilon_1, \ldots, \varepsilon_i)$ for each $i \geq 0$. We assume that $\mathbf{R}_0 = \mathbf{R}$.

For an $\mathbf{R}_i$-finite element $b \in \mathbf{R}_{i+1}$ its standard part (relative to $\mathbf{R}_i$) denote by $\mathrm{st}_i(b) \in \mathbf{R}_i$. For any $b \in \mathbf{R}_j$, $j > i$ we define $\mathrm{st}_i(b) = \mathrm{st}_i(\mathrm{st}_{i+1}(\ldots \mathrm{st}_{j-1}(b) \ldots)$. For a semialgebraic set $V \subset \mathbf{F}_1^n$ defined by a certain formula Φ of the first order theory of the real closed field $\mathbf{F}_1$ and for a real closed field $\mathbf{F}_2 \supset \mathbf{F}_1$ we define the completion $V^{(\mathbf{F}_2)} \subset \mathbf{F}_2^n$ of V as the semialgebraic set given in $\mathbf{F}_2^n$ by the same formula Φ (we say that $V^{(\mathbf{F}_2)}$ is *defined over*

$\mathbf{F}_1$). We omit super-index $(\mathbf{F}_2)$ in $V^{(\mathbf{F}_2)}$ when this does not lead to ambiguity. In a similar way one can define completions of polynomials and algebraic functions.

Observe that for any polynomial $f \in \mathbf{R}_i[X_1, \ldots, X_n]$ and a point $w \in \mathbf{R}_j^n$, $j \geq i$ such that $\mathrm{st}_i(y)$ is definable, we have $\mathrm{st}_i(f(y)) = f(\mathrm{st}_i(y))$.

Denote by $B_x(r)$ the open ball in $\mathbf{R}_i^n$ centered at x and of radius r, and by $\| \cdot \|$ the completion of Euclidean distance function.

The following lemma shows that the standard part of a semialgebraic set coincides with the standard part of its completion.

Lemma 2 *Let $\mathbf{R}_m \subset \mathbf{F} \subset \mathbf{R}_j$ where $\mathbf{F}$ is a real closed field and $V \subset \mathbf{F}^n$ is a semialgebraic set defined over $\mathbf{F}$. Then $\mathrm{st}_m(V) = \mathrm{st}_m(V^{(\mathbf{R}_j)})$.*

Proof. The inclusion

$$\mathrm{st}_m(V) \subset \mathrm{st}_m(V^{(\mathbf{R}_j)})$$

is trivial.

To prove the opposite inclusion take a point $x \in \mathrm{st}_m(V^{(\mathbf{R}_j)})$ and consider a semialgebraic set $\{\|x - y\|^2 : \ y \in V\} \subset \mathbf{F}$. This set is a finite union of (either closed, either open or semi-open) intervals. This is obvious for a semialgebraic subset of $\mathbf{R}$, for an arbitrary real closed field this follows from the transfer principle. Let ω be the left endpoint of the left-most among these intervals. If $x \notin \mathrm{st}_m(V)$ then there exists $0 < r_0 \in \mathbf{R}_m$ such that $\omega > r_0^2$, hence $B_x(r_0) \cap V = \emptyset$. By the transfer principle the completion of the latter set is also empty: $B_x(r_0) \cap V^{(\mathbf{R}_j)} = \emptyset$. This contradicts to the inclusion $x \in \mathrm{st}_m(V^{(\mathbf{R}_j)})$ and proves the lemma.

For a subset $E \subset \mathbf{R}_m^n$ denote by $cl(E)$ its closure in the topology with the base of all open balls. Denote by ∂E the boundary

$$\{y \in \mathbf{R}_m^n : \text{ for any } 0 < r \in \mathbf{R}_m \ \emptyset \neq B_y(r) \cap E \neq B_y(r)\}.$$

Note that the above definition of the closure, being applied to a semialgebraic set and written as a formula of first order theory of the field $\mathbf{R}_m$, involves quantifiers. The following lemma shows that the closure of a semialgebraic set can be described in terms of infinitesimals.

Lemma 3 (cf. Lemma 1 [GV 92]).
a) Let polynomials

$$h_1, \ldots, h_j, g_1, \ldots, g_s \in \mathbf{R}_q[X_1, \ldots, X_n]$$

and natural numbers q, l, m satisfy inequalities $q < l < m$. Consider semialgebraic sets

$$V = \{g_1 \geq 0, \ldots, g_s \geq 0, h_1 > 0, \ldots, h_j > 0\} \subset \mathbf{R}_q^n$$

and

$$\mathcal{V} = \{g_1 > -\varepsilon_m, \ldots, g_s > -\varepsilon_m, h_1 > \varepsilon_l, \ldots, h_j > \varepsilon_l\} \subset \mathbf{R}_m^n.$$

Then

$$cl(V) = \mathrm{st}_q(\mathcal{V}) = \mathrm{st}_q(cl(\mathcal{V})).$$

b) $\partial V \subset \mathrm{st}_q(\partial\mathcal{V})$.

Proof.

a) Let $x \in cl(\mathcal{V})$ and the standard part $y = \mathrm{st}_q(x)$ be definable. We prove that $y \in cl(V)$. Consider a point $y_1 = \mathrm{st}_l(x)$, then

$$g_{s_1}(y_1) = \mathrm{st}_l(g_{s_1}(x)) \geq 0,\ 1 \leq s_1 \leq s;\ h_{j_1}(y_1) = \mathrm{st}_l(h_{j_1}(x)) \geq \varepsilon_l,\ 1 \leq j_1 \leq j.$$

Hence $y_1 \in V^{(\mathbf{R}_l)}$.

If $y \notin cl(V)$ then there exists $0 < r \in \mathbf{R}_q$ such that $B_y(r) \cap V = \emptyset$. Due to the transfer principle the latter relation holds also over the field $\mathbf{R}_l$, namely, $B_y(r) \cap V^{(\mathbf{R}_l)} = \emptyset$. On the other hand, $y_1 \in B_y(r) \cap V^{(\mathbf{R}_l)}$ since $\mathrm{st}_q(y_1) = y$. The obtained contradiction proves the inclusion $\mathrm{st}_q(cl(\mathcal{V})) \subset cl(V)$.

Now let $y \in cl(V)$. Consider a semialgebraic set $\{\|y - z\|^2 :\ z \in \mathcal{V}\} \subset \mathbf{R}_m$. Then this set is a finite union of (either closed, either open or semi-open) intervals (cf. the proof of Lemma 2). Let ω be the left endpoint of the left-most among these intervals. If $y \notin \mathrm{st}_q(\mathcal{V})$ then there exists an element r_1, $0 < r_1 \in \mathbf{R}_q$ such that $\omega > r_1^2$, i.e., $\mathcal{V} \cap B_y(r_1) = \emptyset$. On the other hand, $V \cap B_y(r_1) \neq \emptyset$ since $y \in cl(V)$. Taking into the account the inclusion $V \subset \mathcal{V}$, we get a contradiction which proves the inclusion $cl(V) \subset \mathrm{st}_q(\mathcal{V})$.

b) Let $x \in \partial V$ and $x \notin \mathrm{st}_q(\partial\mathcal{V})$. Then there exists an element r_2, $0 < r_2 \in \mathbf{R}_q$ such that $B_x(r_2) \cap \partial\mathcal{V} = \emptyset$ (cf. the proof of a)). Because of a), $x \in \mathrm{st}_q(\mathcal{V})$, therefore $B_x(r_2) \subset \mathcal{V}$. On the other hand, $\mathcal{V} \cap \mathbf{R}_q^n \subset V$, hence $B_x(r_2) \cap \mathbf{R}_q^n \subset V$, this contradicts to the inclusion $x \in \partial V$.

Lemma is proved.

In the proof of Lemma 3 a) it was actually shown that for any semialgebraic set $U \subset \mathbf{R}_m^n$ we have $\mathrm{st}_q(U) = \mathrm{st}_q(cl(U))$, $q < m$.

Corollary. *Denote*

$$V_0 = \{h = 0, h_1 > 0, \ldots, h_j > 0\} \subset \mathbf{R}_q^n,$$

$$\mathcal{V}_0 = \{h = \varepsilon_m, h_1 > \varepsilon_l, \ldots, h_j > \varepsilon_l\} \subset \mathbf{R}_m^n.$$

Then $\mathrm{st}_q(\mathcal{V}_0) \subset cl(V_0)$.

To prove Corollary, in Lemma 3 a) instead of $\mathcal{V}$ consider a modified set

$$\{-2\varepsilon_m < h < 2\varepsilon_m, h_1 > \varepsilon_l, \ldots, h_j > \varepsilon_l\} \supset \mathcal{V}_0.$$

Lemma 4 (cf. Lemma 4a) in [GV 88]). *Let F be a smooth algebraic function defined on an open semialgebraic set $U \subset \mathbf{R}_i^n$ and determined by a polynomial with coefficients* *from $\mathbf{R}_i$. Then ε_{i+1} is not a critical value of F (i.e., $\mathrm{grad}_y(F)$ does not vanish at any point $y \in \{F = \varepsilon_{i+1}\} \cap U$).*

Proof. Sard's theorem [Hi 76] and the transfer principle imply the finiteness of the set of all critical values of F, moreover this set lies in $\mathbf{R}_i$.

3. Curved points

For any i-face P_i denote by $\overline{P}_i$ the i-plane containing P_i.

First let us reduce Theorem 2 to the case of compact P. Let t be the minimal dimension of faces of P and P_t be a face with $\dim(P_t) = t$. Then $\overline{P}_t$ is a t-plane.

For each i-face P_i of P with $\dim(P_i \cap W) = i$ choose a point $x_{P_i} \in (P_i \cap W)$ such that a suitable neighbourhood of x_{P_i} in P_i is contained in W.

First consider the case $t \geq 1$. Choose any hyperplane Σ transversal to P_t such that the points x_{P_i} for all i-faces P_i lie in one of two semi-spaces of $\mathbf{R}^n \setminus \Sigma$, denote this semi-space by $\tilde{\Sigma}$. Replace P by $(P \cap \tilde{\Sigma}) \cup \Sigma$ reducing t by one. Continue this process while $t \geq 1$.

Now consider the case $t = 0$.

Observe that there exists a linear form $L = \tau_1 X_1 + \cdots + \tau_n X_n$ with $\tau_j \in \mathbf{R}$, $1 \leq j \leq n$ such that for every $\gamma \in \mathbf{R}$ the intersection $\{L + \gamma \geq 0\} \cap P$ is compact. Take γ such that $x_{P_i} \in P' = \{L + \gamma \geq 0\} \cap P$ for all P_i. The number of all i-faces P'_i of P' such that $\dim(P'_i \cap W) = i$ is greater or equal than the number of all i-faces P_i of P such that $\dim(P_i \cap W) = i$. ¿From now on we assume, without loss of generality, that P is compact.

For an m-plane $Q \subset \mathbf{R}^n_m$ and a point $x \in \mathbf{R}^n_m$ denote by $Q(x)$ the m-plane collinear to Q and containing x.

Two planes Q_1, Q_2 of arbitrary dimensions are called *transversal* if

$$\dim\bigl(Q_1(0) \cap Q_2(0)\bigr) = \max\{0,\ \dim\bigl(Q_1(0)\bigr) + \dim\bigl(Q_2(0)\bigr) - n\}.$$

For every $0 \leq i < n$ choose an $(n-i)$-plane Π_{n-i} (defined over $\mathbf{R}$) transversal to any face of the polyhedron P.

Denote $f = f_1^2 + \cdots + f_{k_1}^2$.

Fix $0 \leq i < n$ and denote by $f^{(x)}$ the restriction of f on $\Pi_{n-i}(x)$ (for $x \in \mathbf{R}^n_m$).

Definition. *A point $y \in \{f = \varepsilon_3\}$ is called i-curved if $\mathrm{grad}_y(f^{(y)} - \varepsilon_3) \neq 0$, all principal curvatures of the variety $\{f^{(y)} = \varepsilon_3\} \subset \Pi_{n-i}(y)$ at y are greater than ε_2^{-1} and $f_{k_1+1}(y) > \varepsilon_2, \ldots, f_k(y) > \varepsilon_2$.*

Remark. We fix an orthogonal basis in $\Pi_{n-i}(0)$ with coordinates belonging to $\mathbf{R}$. Then in Definition we consider curvatures in $\Pi_{n-i}(y)$ with respect to the basis obtained from the fixed one by the shift $Y \longrightarrow Y + y$.

One can consider this definition as a kind of "localization" of the key concept of an *angle point* from [GV 94].

Denote the set of all i-curved points by $\mathcal{K}_i \subset \mathbf{R}^n_3$. Observe that $\mathcal{K}_i$ is semialgebraic due to the remark at the end of Section 1. Denote $K_i = \mathrm{st}_0(\mathcal{K}_i) \subset \mathbf{R}^n$, this set is closed semialgebraic by Lemma 5.1 from [RV 94]. Corollary to Lemma 3 implies that $K_i \subset cl(W)$.

Lemma 5. *Let for an i-facet P_i of P the dimension* $\dim(W \cap P_i) = i$. *Then* $W \cap P_i \subset K_i$.

Proof. Let $x \in W \cap P_i$. Then $f_j(x) > c$, $k_1 + 1 \leq j \leq k$ for a certain $0 < c \in \mathbf{R}$. Hence there exists $0 < r \in \mathbf{R}$ such that for any point y from the open ball $B_x(r)$ we have $f_j(y) > c$, $k_1 + 1 \leq j \leq k$. Due to the transfer principle, $f_j(y) > c$, $k_1 + 1 \leq j \leq k$ for any point $y \in B_x(r) \cap \mathbf{R}_3^n$.

Observe that $\{f = \varepsilon_3\} \cap \Pi_{n-i}(x) = \{f^{(x)} = \varepsilon_3\}$ is a smooth hypersurface in $\Pi_{n-i}(x)$, because $x \in \mathbf{R}^n$ and ε_3 is not a critical value of the polynomial $f^{(x)}$ by Lemma 4.

Our purpose is to prove that $x = \mathrm{st}_2(y)$ (and *a fortiori* $x = \mathrm{st}_0(y)$) for a suitable $y \in \{f = \varepsilon_3\} \cap \Pi_{n-i}(x)$ such that all principal curvatures of the variety $\{f = \varepsilon_3\} \cap \Pi_{n-i}(x)$ at the point y are greater than ε_2^{-1}. This would imply Lemma 5 since $\mathrm{grad}_y(f^{(x)} - \varepsilon_3) \neq 0$ (see Definition).

The point x is a vertex of the polyhedron $\mathcal{P} = P \cap \Pi_{n-i}(x)$ because Π_{n-i} is transversal to P_i. Note that for each $(n-i-1)$-face of $\mathcal{P}$ the normalized orthogonal vector (in $\Pi_{n-i}(x)$) to this face has all coordinates in $\mathbf{R}$. The vertex x belongs to at least $(n-i)$ among $(n-i-1)$-faces of the polyhedron $\mathcal{P}$.

Choose any $(n-i-1)$-faces $T_1, \dots, T_{n-i}$ of this kind. Denote by $\mathcal{T} \subset \Pi_{n-j}(x)$ the closed cone with vertex at x, formed by planes $\overline{T}_1, \dots, \overline{T}_{n-i}$ and containing $\mathcal{P}$.

Observe that for any point $y \in cl(B_x(r/2)) \cap \mathbf{R}_3^n$ the inequalities $f_j(y) > c$, $k_1 + 1 \leq j \leq k$ hold, since $cl(B_x(r/2)) \subset B_x(r)$. Therefore,

$$\{f = 0, f_j > c,\ k_1 + 1 \leq j \leq k\} \cap cl(B_x(r/2)) = \{f = 0\} \cap cl(B_x(r/2)).$$

Denote $D = \mathcal{T} \cap cl(B_x(r/2))$. For any point $z \in \{f = \varepsilon_3\} \cap \Pi_{n-i}(x) \cap cl(B_x(r/2))$ we have $\mathrm{st}_2(z) \in \{f = 0\} \cap \Pi_{n-i}(x) \cap cl(B_x(r/2))$ (see Section 2). Hence

$$\mathrm{st}_2(z) \in W \cap \Pi_{n-i}(x) \cap cl(B_x(r/2)) \subset \mathcal{P} \cap cl(B_x(r/2)) \subset D,$$

in particular, the distance $\rho(z, D)$ from z to D is infinitesimal relative to $\mathbf{R}_2$ (the distance from a point to a bounded set closed in the topology with the base of all open balls, does exist because it is true over the field $\mathbf{R}$, over arbitrary real closed field use the transfer principle).

Since the set $\{f = \varepsilon_3\} \cap \Pi_{n-i}(x) \cap cl(B_x(r/2))$ is bounded and closed in the topology with the base of all open balls, there exists $\rho_0 = \max \rho(z, D)$ (sf. the above arguing) where z ranges over all points from $\{f = \varepsilon_3\} \cap \Pi_{n-i}(x) \cap cl(B_x(r/2))$, and ρ_0 is infinitesimal relative to $\mathbf{R}_2$.

Let us shift (in Π_{n-i}) each $(n-i-1)$-dimensional plane among $\overline{T}_1, \dots, \overline{T}_{n-i}$ parallel to itself outward the cone $\mathcal{T}$ to the distance ρ_0. The shifted planes form a new closed cone $\mathcal{T}'$ with a vertex x'. Obviously $\mathcal{T} \subset \mathcal{T}'$. Observe that the distance $\|x - x'\|$ is infinitesimal relative to $\mathbf{R}_2$. Denote $D' = \mathcal{T}' \cap cl(B_x(r/2))$. Then $\{f = \varepsilon_3\} \cap \Pi_{n-i}(x) \cap cl(B_x(r/2)) \subset D'$.

In the plane $\Pi_{n-i}(x)$ choose a hyperplane Q such that the coordinates of the normalized vector orthogonal to Q belong to $\mathbf{R}$, and $\mathcal{T}' \cap Q(x') = \{x'\}$. Take a hyperplane $Q(y)$ (in $\Pi_{n-i}(x)$) such that its distance from the point x' is positive, belongs to $\mathbf{R}$, and $\mathcal{T}' \cap Q(y) \neq \emptyset$. Observe that $\mathcal{T}' \cap Q(y)$ is contained in a certain $(n-i-1)$-dimensional open ball $B \subset Q(y)$ with the center z such that the radius and the coordinates of the point $z - x'$ belong to $\mathbf{R}$.

There exists the unique $(n-i-1)$-dimensional sphere $\mathcal{S} \subset \Pi_{n-i}(x)$ containing both the point $(x'+z)/2$ and the $(n-i-2)$-dimensional sphere ∂B. Then the point x' lies outside the $(n-i)$-dimensional open ball $\mathcal{B}$ bounded by $\mathcal{S}$.

Denote by $\mathcal{T}'' \subset \Pi_{n-i}(x)$ the closed cone with the vertex at x' and with the base $\mathcal{S}$. Then $\mathcal{T}' \subset (\mathcal{T}'' \setminus \partial\mathcal{T}'')$ because $\mathcal{T}' \cap Q(y) \subset B \subset \mathcal{B}$.

The intersection $\mathcal{S} \cap \partial\mathcal{T}''$ is a $(n-i-2)$-dimensional sphere situated in a certain hyperplane Γ (in $\Pi_{n-i}(x)$). Then $\mathcal{S} \cap \partial\mathcal{T}''$ divides $\mathcal{S} \setminus (\mathcal{S} \cap \partial\mathcal{T}'')$ into two connected components $\mathcal{S}_1$ and $\mathcal{S}_2$. Let $\mathcal{S}_1$ be located in the same half-space (in $\Pi_{n-i}(x)$) with the boundary Γ as the point x'.

Denote by $\mathcal{S}_1(\mu)$ the dilation of $\mathcal{S}_1$ with the coefficient μ with respect to the point x'. Observe that the open cone $\mathcal{T}'' \setminus \partial\mathcal{T}''$ is the disjoint union of the dilations $\mathcal{S}_1(\mu)$ over all $0 < \mu \in \mathbf{R}_3$. There exists the minimal $\mu_0 > 0$ such that

$$\mathcal{S}_1(\mu_0) \cap \{f = \varepsilon_3\} \cap \Pi_{n-i}(x) \cap cl(B_x(r/2)) \neq \emptyset.$$

Then $\mathcal{S}_1(\mu_0)$ divides the open cone $\mathcal{T}'' \setminus \partial\mathcal{T}''$ into two connected components, moreover the set

$$\{f = \varepsilon_3\} \cap \Pi_{n-i}(x) \cap cl(B_x(r/2))$$

and all points from $\mathcal{T}'' \setminus \partial\mathcal{T}''$ sufficiently close to the point x' belong to the different connected components.

Taking into the account that $f(x) = 0$, and applying Lemma 3 from [GV 88] to the polynomial $f^{(x)}$, we conclude that there exists a point $y_0 \in \{f = \varepsilon_3\} \cap \Pi_{n-i}(x)$ such that the distance $\|x - y_0\|$ is infinitesimal relative to $\mathbf{R}_2$. Evidently, $y_0 \in cl(B_x(r/2))$ and $\|y_0 - x'\|$ is also infinitesimal relative to $\mathbf{R}_2$. Hence μ_0 is infinitesimal relative to $\mathbf{R}_2$ as well. Therefore, the radius ν of the sphere $\mathcal{S}(\mu_0)$ is also infinitesimal relative to $\mathbf{R}_2$.

Consider a point

$$y \in \mathcal{S}_1(\mu_0) \cap \{f = \varepsilon_3\} \cap \Pi_{n-i}(x) \cap cl(B_x(r/2)).$$

Then $\|y - x'\|$ is infinitesimal relative to $\mathbf{R}_2$. Besides, the hypersurfaces $\mathcal{S}_1(\mu_0)$ and $\{f = \varepsilon_3\} \cap \Pi_{n-i}(x)$ (as well as the set $\{f = \varepsilon_3\} \cap \Pi_{n-i}(x) \cap cl(B_x(r/2))$) has the same tangent plane T (in $\Pi_{n-i}(x)$) at the point y.

Let $\mathcal{H}_y$ be $(n-i-1) \times (n-i-1)$-matrix introduced in Section 1 (with $f^{(x)}$ playing the role of F and y playing the role of x). For any normalized vector $v \in T$ the second derivative $v\mathcal{H}_y v$ of the function H_y (see Section 1) in the direction v is greater or equal to the corresponding second derivative for the sphere $\mathcal{S}(\mu_0)$ (at the point y). The latter second derivative equals to $1/\nu$ (cf. the proof of Theorem 4 in Ch. 12 [Th 77]). In particular, for the principal curvatures of the hypersurface $\{f^{(x)} = \varepsilon_3\} = \{f = \varepsilon_3\} \cap \Pi_{n-i}(x)$ (in $\Pi_{n-i}(x)$), the inequalities $\lambda_2 \geq 1/\nu, \ldots, \lambda_{n-i} \geq 1/\nu$ are valid, hence $\lambda_2 > \varepsilon_2^{-1}, \ldots, \lambda_{n-i} > \varepsilon_2^{-1}$.

Thus, the point y is i-curved (recall that $f_j(y) > c > \varepsilon_2$, $k_1 + 1 \leq j \leq k$ since $y \in B_x(r)$).

Finally, $\mathrm{st}_2(y) = x$, because $\|x - y\|$ is infinitesimal relative to $\mathbf{R}_2$ and $x \in \mathbf{R}^n$, *a fortiori* $\mathrm{st}_0(y) = x$, i.e., $x \in K_i$. Lemma is proved.

Corollary. *If* $\dim(W \cap P_i) = i$ *then* $\dim(K_i \cap P_i) = i$.

This Corollary implies that in order to prove Theorem 2 it is sufficient to bound the number of i-facets P_i for which $\dim(K_i \cap P_i) = i$.

Lemma 6. *For any smooth point $z \in K_i$ with the dimension $\dim_z(K_i) \geq i+1$ the tangent plane T_z to K_i at z is not transversal to Π_{n-i}.*

Remark. In the particular case $i = 0$ Lemma 6 states that K_0 consists of a finite number of points.

Proof of Lemma 6. First let us reduce the proof to the case $i = 0$ (so assume in the reduction that $\dim(K_0) \leq 0$). Thus, let $i \geq 1$ and suppose that $e = \dim_z(K_i) \geq i+1$. Assume that T_z is transversal to Π_{n-i}, then $\dim(T_z \cap \Pi_{n-i}(z)) = e - i$. Take any $(n-e)$-plane $R \subset \Pi_{n-i}(z)$ defined over $\mathbf{R}$ for which $T_z \cap R = \{z\}$. Consider the linear orthogonal projection $\pi : \mathbf{R}_3^n \longrightarrow \mathbf{R}_3^e$ onto e-subspace along R. Then $\dim(\pi(T_z)) = e$. Therefore, by the implicit function theorem, $\pi(K_i) \subset \mathbf{R}^e$ contains e-dimensional ball $B_{\pi(z)}(r)$ for a certain $0 < r \in \mathbf{R}$.

For any point $x \in K_i$ there is a point $x' \in \mathcal{K}_i$ such that $\mathrm{st}_0(x') = x$, hence $\mathrm{st}_0(\pi(\mathcal{K}_i)) \supset B_{\pi(z)}(r)$.

For any point $y \in \mathbf{R}^n$ the set $\mathcal{K}_0^{(y)}$ of 0-curved points of the restriction $f^{(y)}$ coincides with $\Pi_{n-i}(y) \cap \mathcal{K}_i$ according to Definition. Applying the assumption that the lemma is valid for $i = 0$ to the polynomial $f^{(y)}$ we obtain the inequality $\dim\big(\mathrm{st}_0(\Pi_{n-i}(y) \cap \mathcal{K}_i)\big) \leq 0$ (taking into the account that $f^{(y)}$ is defined over $\mathbf{R}$).

Let us show that $\pi(\mathcal{K}_i)$ does not contain a ball $B_w(r_1)$ for any $0 < r_1 \in \mathbf{R}$ and $w \in \mathbf{R}_3^n$. Assume the contrary, then there exists a point $w_1 \in B_w(r_1) \cap \mathbf{R}^e$. Let $y_1 \in \mathbf{R}^n$ be a point such that $\pi(y_1) = w_1$. Denote $\Pi' = \pi(\Pi_{n-i})$, then $\dim(\Pi') = e - i$, $\Pi_{n-i} = \pi^{-1}(\Pi')$. Then the following inequalities hold:

$$\dim \mathrm{st}_0\big(\Pi'(w_1) \cap \pi(\mathcal{K}_i)\big) \geq \dim \mathrm{st}_0\big(\Pi'(w_1) \cap B_w(r_1)\big) = e - i \geq 1.$$

On the other hand, $\Pi'(w_1) \cap \pi(\mathcal{K}_i) = \pi(\mathcal{K}_i \cap \Pi_{n-i}(y_1))$, and, therefore,

$$\dim \mathrm{st}_0\big(\Pi'(w_1) \cap \pi(\mathcal{K}_i)\big) \leq \dim \mathrm{st}_0\big(\mathcal{K}_i \cap \Pi_{n-i}(y_1)\big) \leq 0,$$

(the latter inequality was proved above). The obtained contradiction shows that $\pi(\mathcal{K}_i)$ does not contain a ball $B_w(r_1)$ for any $0 < r_1 \in \mathbf{R}$.

We claim that for any ball $B_{y_2}(r_2) \subset B_{\pi(z)}(r)$ defined over $\mathbf{R}_3$ such that $0 < r_2 \in \mathbf{R}$, the intersection $B_{y_2}(r_2) \cap \partial\pi(\mathcal{K}_i) \neq \emptyset$. Assume the contrary. Then either $B_{y_2}(r_2) \subset \pi(\mathcal{K}_i)$ or $B_{y_2}(r_2) \cap \pi(\mathcal{K}_i) = \emptyset$. The inclusion $B_{y_2}(r_2) \subset \pi(\mathcal{K}_i)$ is impossible as was shown above. If $B_{y_2}(r_2) \cap \pi(\mathcal{K}_i) = \emptyset$, then $\mathrm{st}_0(y_2) \notin \mathrm{st}_0(\pi(\mathcal{K}_i))$, the latter contradicts the inclusions $B_{\mathrm{st}_0(y_2)}(r_2/2) \subset B_{\pi(z)}(r) \subset \mathrm{st}_0(\pi(\mathcal{K}_i))$ of the sets in the space $\mathbf{R}^e$. This proves the claim.

Observe that $\dim\big(\partial(\pi(\mathcal{K}_i))\big) \leq e - 1$. Applying Lemma 5.1 from [RV 94], we get $\dim \mathrm{st}_0\big(\partial(\pi(\mathcal{K}_i))\big) \leq e - 1$.

On the other hand we shall now prove that $\mathrm{st}_0\big(\partial(\pi(\mathcal{K}_i))\big) \supset B_{\pi(z)}(r)$. This contradiction would complete the proof of the reduction of the lemma to the case $i = 0$. Indeed, let $y_3 \in B_{\pi(z)}(r)$. Observe that the set $\{\|y - y_3\|^2 : y \in \partial(\pi(\mathcal{K}_i))\}$ is semialgebraic. Hence, this set is a finite union of points and intervals (cf. the proof of Lemma 2). Let ω be the minimal among these points and the endpoints of these intervals. Suppose that $y_3 \notin \mathrm{st}_0\big(\partial(\pi(\mathcal{K}_i))\big)$, i.e., there does not exist $y \in \partial(\pi(\mathcal{K}_i))$ such that $\mathrm{st}_0(y) = y_3$. Thus, $\omega > r_3^2$ for a suitable $0 < r_3 \in \mathbf{R}$. It follows that $B_{y_3}(r_3) \cap \partial(\pi(\mathcal{K}_i)) = \emptyset$. This contradicts to the proved above claim.

Now let $i = 0$. Suppose that the statement of the lemma is wrong and $\dim(K_0) = s \geq 1$. There is a linear projection $\phi : \mathbf{R}_3^n \longrightarrow \mathbf{R}_3$ onto one of the coordinates such that $\phi(K_0) \supset [\eta_1', \eta_2']$ for some $\eta_1',\ \eta_2' \in \mathbf{R}$, $\eta_1' < \eta_2'$. Since $\mathrm{st}_0(\phi(\mathcal{K}_0)) \supset [\eta_1', \eta_2']$ and $\phi(\mathcal{K}_0) \subset \mathbf{R}_3$, being a semialgebraic set, consists of a finite union of intervals and points, there exist $\eta_1,\ \eta_2 \in \mathbf{R}$, $\eta_1 < \eta_2$ such that $\phi(\mathcal{K}_0) \supset [\eta_1, \eta_2]^{(\mathbf{R}_3)}$.

Our nearest purpose is to prove the existence of a semialgebraic curve $C' \subset \mathcal{K}_0$ such that the mapping $\phi :\ C' \longrightarrow [\eta_1, \eta_2]^{(\mathbf{R}_3)}$ is bijective.

For any point $u \in [\eta_1, \eta_2]^{(\mathbf{R}_3)}$ take the unique point $v_u \in \mathcal{K}_0$ such that $\phi(v_u) = u$ according to the following rule (which is, in fact, quite flexible).

A projection $\pi_1(\phi^{-1}(u))$ of $\phi^{-1}(u)$ onto the coordinate X_1, being a semialgebraic set, is a union of a finite number of points and intervals (with or without endpoints). Let μ_1, μ_2 be the endpoints of the left-most interval.

Consider four cases. In the first case $\mu_1,\ \mu_2 \in \mathbf{R}_3$, then put $\mu = (\mu_1 + \mu_2)/2$. In the second case the interval is given either by inequality $X < \mu_2$ or by inequality $X \leq \mu_2$, we put $\mu = \mu_2 - 1$. In the third case the interval is either $\{X > \mu_1\}$ or $\{X \geq \mu_1\}$, we put $\mu = \mu_1 + 1$. In the last case the interval coincides with the whole $\mathbf{R}_3$, we put $\mu = 0$.

Note that $\mu \in \pi_1(\phi^{-1}(u))$. We fix the first coordinate of the point v_u under construction equal to μ.

Consider the projection $\pi_2(\phi^{-1}(u) \cap \{X_1 = \mu\})$ onto the axis X_2. Continuing in the similar way, after n steps we obtain a point $v_u = (\mu, \dots) \in \phi^{-1}(u)$.

We define the semialgebraic curve C' to be the set of all the obtained points v_u for $u \in [\eta_1, \eta_2]^{(\mathbf{R}_3)}$.

The curve C' has only a finite number of singular points (this is well-known for algebraic curves over $\mathbf{R}$, for arbitrary real closed fields we use the transfer principle). The curve C' with deleted singular points is a finite union of smooth connected semialgebraic curves. Take one of these curves C such that $\phi(C) \supset [\eta_3, \eta_4]^{(\mathbf{R}_3)}$ for appropriate $\eta_3 < \eta_4$, $\eta_3,\ \eta_4 \in \mathbf{R}$.

Since $C \subset \mathcal{K}_0$, Theorem 1 from Ch. 9 in [Th 77] implies that for any point $w \in C$ its curvature $k(w)$ is greater or equal to the minimum of principal curvatures of the hypersurface $\{f = \varepsilon_3\}$ at this point w, hence $k(w) > \varepsilon_2^{-1}$ (according to Definition).

Consider the Gauss map $\mathcal{G} :\ C \longrightarrow \mathcal{S}^{n-1}$ where $\mathcal{S}^{n-1}$ is $(n-1)$-sphere and for a point $w \in C$ the image $\mathcal{G}(w)$ is the normalized vector tangent to C at w.

Let us prove the following statement.

For any reals θ, l and any smooth semialgebraic curve $\mathcal{C}$ with the projection on a certain coordinate axis greater than l and with the curvature at each point greater than θ, there exists a hyperplane Δ such that the semialgebraic set $\Delta \cap \mathcal{S}^{n-1} \cap \mathcal{G}(\mathcal{C})$ has the dimension zero and contains at least $\lfloor l\theta/\pi \rfloor$ points. To prove this statement for a curve $\mathcal{C}$ defined over $\mathbf{R}$ observe that the length (with multiplicities) of the image $\mathcal{G}(\mathcal{C}) \subset \mathcal{S}^{n-1}$

equals to

$$\int_{w \in \mathcal{C}} k(w) \geq l\theta$$

(cf. Ch. 10 in [Th 77]). Observe that the length of a curve $\mathcal{C}_1 \subset \mathcal{S}^{n-1}$ equals to the average (with respect to the uniform Borel measure) number of points of intersection $\mathcal{C}_1 \cap \mathcal{S}^{n-1} \cap \Delta$ over all hyperplanes Δ, multiplied by π. This implies the statement for the semialgebraic curves $\mathcal{C}$ defined over $\mathbf{R}$. For curves $\mathcal{C}$ defined over an arbitrary real closed field this statement follows from the transfer principle (applied for fixed θ and l).

Applying the statement to the curve C with $l = \eta_4 - \eta_3$ and fixed arbitrary real θ (taking into the account that for any point $w \in C$ the curvature $k(w) > \varepsilon_2^{-1} > \theta$), we conclude that there exists a vector $(\rho_1, \dots, \rho_n)$ such that C contains at least $\lfloor l\theta/\pi \rfloor$ points w_1 with the tangent vector t_{w_1} to C at w_1 satisfying the linear equation $t_{w_1} \cdot (\rho_1, \dots, \rho_n) = 0$, and there is a finite number of such points.

One can formulate the condition $t_{w_1} \cdot (\rho_1, \dots, \rho_n) = 0$ on a point $w_1 \in C$ as a formula of the first-order theory of real closed fields (for a fixed θ). Therefore, there is only a fixed finite number (depending on C) of such points w_1, but since one can take an arbitrary θ, we get a contradiction.

This implies that $\dim(K_0) \leq 0$ and completes the proof of the lemma.

4. Facets of P and Whitney stratification of K_i

Recall that K_i, as any semialgebraic set, admits a Whitney stratification (see, e.g., [GM 88]). Namely, K_i can be represented as a disjoint union $K_i = \bigcup_j S_j$ of a finite number of semialgebraic sets, called strata, which are smooth manifolds and such that:

(1) (frontier condition) $S_{j_1} \cap cl(S_{j_2}) \neq \emptyset$ if and only if $S_{j_1} \subset cl(S_{j_2})$ (this defines a partial order $S_{j_1} \prec S_{j_2}$ on the strata);

(2) (Whitney condition A) Let $S_{j_1} \subset cl(S_{j_2})$ and a sequence of points $y_m \in S_{j_2}$ tends to a point $y \in S_{j_1}$ when $m \to \infty$. Assume that the sequence of tangent planes T_{y_m} to S_{j_2} at points y_m tends to a certain plane T. Then $T_y \subset T$ where T_y is a tangent plane to S_{j_1} at y.

Lemma 7. *Let for an i-face P_i of P the dimension $\dim(K_i \cap P_i) = i$. Assume that S'_j is a connected component of a stratum S_j of K_i such that $\dim\big(cl(S'_j) \cap K_i \cap P_i\big) = i$. Then $S'_j \subset P_i$.*

Proof. If $\dim(S_j) = i$ then $S'_j \subset P_i$ because $S'_j \subset K_i \subset cl(W)$ (see the definition of K_i in Section 3) and $cl(W) \subset P$, taking into the account that S'_j is a connected smooth semialgebraic set.

Now let $e = \dim S_j \geq i + 1$. We can assume without loss of generality that S_j is one of the maximal strata (with respect to the partial order $\prec$), otherwise take a maximal stratum containing S_j in its closure.

There is a stratum S_l such that $\dim(S_l \cap cl(S'_j) \cap K_i \cap P_i) = i$. The property (1) of Whitney stratification implies that $S_l \subset cl(S_j)$. Take a connected component S'_l of S_l for which $\dim(S'_l \cap cl(S'_j) \cap K_i \cap P_i) = i$. Then $\dim(S'_l) = i$, i.e., $\dim(S_l) = i$ because S'_l is smooth and $S'_l \subset P$, hence $S'_l \subset P_i$ arguing as above. Let a point $y \in S'_l \cap cl(S'_j) \cap K_i \cap P_i$ be such that for a suitable $0 < r \in \mathbf{R}$ we have

$$(B_y(r) \cap P_i) \subset (S'_l \cap cl(S'_j) \cap K_i \cap P_i),$$

then $T_y(S'_l) = \overline{P}_i$.

There exists a converging sequence $y_m \longrightarrow_{m \to \infty} y$, $y_m \in S'_j$ such that the sequence of e-dimensional tangent planes $T_{y_m}(S'_j)$ converges when $m \to \infty$ to a certain e-dimensional plane τ. Due to (2) (Whitney condition A), $\overline{P}_i \subset \tau$.

Lemma 6 implies that $T_{y_m}(S'_j)$ is not transversal to Π_{n-i} (taking into the account that y_m is a smooth point of K_i because S_j is a maximal stratum). Therefore, $e'_m = \dim(T_{y_m}(S'_j) \cap \Pi_{n-i}(y_m)) \geq e - i + 1$. Some subsequence $T_{y_{m_q}}(S'_j) \cap \Pi_{n-j}(y_{m_q})$ of planes converges when $q \to \infty$ to a certain e'-dimensional plane $\chi \subset \Pi_{n-i}(y)$, where $e' = e'_{m_q} \geq e - i + 1$ for any large enough q.

Choose a basis $a_1, \dots, a_i$ of i-plane $\overline{P}_i(0)$ and a basis $b_1, \dots, b_{e'}$ of $\chi(0)$. Then vectors $a_1, \dots, a_i, b_1, \dots, b_{e'}$ are linearly independent due to transversality of $\overline{P}_i$ and Π_{n-i}. For large enough q_0, for any $q \geq q_0$ there exist vectors

$$a_1^{(q)}, \dots, a_i^{(q)}, b_1^{(q)}, \dots, b_{e'}^{(q)} \in \big(T_{y_{m_q}}(S'_j)\big)(0)$$

situated sufficiently close to vectors $a_1, \dots, a_i, b_1, \dots, b_{e'}$, respectively, so that the vectors $a_1^{(q)}, \dots, a_i^{(q)}, b_1^{(q)}, \dots, b_{e'}^{(q)}$ are also linearly independent. Hence $\dim(T_{y_{m_q}}(S'_j)) \geq e' + i \geq e + 1$. This leads to a contradiction with the equality $\dim(S_j) = e$ and proves the lemma.

Denote $g = f_{k_1+1} \cdots f_k$. Choose $0 < \mu \in \mathbf{R}$ satisfying the following properties:

(a) μ is less than the absolute values of all critical values of the restrictions of g on i-facets P_i (note that Sard's theorem implies the finiteness of the number of all critical values, moreover they all belong to $\mathbf{R}$ due to Lemma 4);

(b) for any P_i such that $\dim(K_i \cap P_i) = i$ the dimension

$$\dim\big(\{g = \mu\} \cap cl(S'_j) \cap K_i \cap P_i\big) \leq i - 2$$

for every connected component S'_j of a stratum S_j such that S'_j is not contained in P_i. Observe that due to Lemma 7 there exists at most finite number of μ violating this condition because $\dim(cl(S'_j) \cap K_i \cap P_i) \leq i - 1$, together with (a) this justifies the existence of the required μ.

Denote $K'_i = K_i \cap \{g = \mu\}$.

Lemma 8.

$$K'_i = \mathrm{st}_0(\mathcal{K}_i \cap \{|g - \mu| < \varepsilon_1\}).$$

Proof. First prove the inclusion $\subset$.

Denote by $\mathbf{F}$ the real closure of the field $\mathbf{R}(\varepsilon_2, \varepsilon_3)$. Since $\mathcal{K}_i$ is defined over $\mathbf{F}$ we have $\mathcal{K}_i = (\mathcal{K}_i \cap \mathbf{F}^n)^{(\mathbf{R}_3)}$. Apply Lemma 2 to the fields $\mathbf{R} \subset \mathbf{F} \subset \mathbf{R}_3$ taking the set $\mathcal{K}_i \cap \mathbf{F}^n$ as V. Then $\mathrm{st}_0(\mathcal{K}_i \cap \mathbf{F}^n) = \mathrm{st}_0(\mathcal{K}_i) = K_i$.

Let $x \in K_i'$. It follows that there exists a point $y \in \mathcal{K}_i \cap \mathbf{F}^n$ such that $\mathrm{st}_0(y) = x$. Hence $\mathrm{st}_0(g(y)) = g(\mathrm{st}_0(y)) = g(x) = \mu$. Then $(g(y) - \mu) \in \mathbf{F}$ is infinitesimal relative to $\mathbf{R}$. Taking into the account the representation of $g(y) - \mu$ as a Puiseux series in ε_3 with the coefficients being, in their turn, Puiseux series in ε_2 (see Section 2), we deduce that $|g(y) - \mu| < \varepsilon_1$. Thus $y \in \mathcal{K}_i \cap \{|g - \mu| < \varepsilon_1\}$, which proves the inclusion $\subset$.

To prove the inclusion $\supset$ take a point $x \in \mathrm{st}_0(\mathcal{K}_i \cap \{|g - \mu| < \varepsilon_1\})$. Then, in particular, $x \in \mathrm{st}_0(\mathcal{K}_i) = K_i$. There exists a point $y \in \mathcal{K}_i \cap \{|g - \mu| < \varepsilon_1\}$ such that $\mathrm{st}_0(y) = x$. Then $\mu = \mathrm{st}_0(g(y)) = g(\mathrm{st}_0(y)) = g(x)$. The lemma is proved.

Lemma 9. *Let for an i-face P_i of P the dimension $\dim(W \cap P_i) = i$. The following equality of the varieties holds:*

$$K_i' \cap P_i = \{g = \mu\} \cap \{f_{k_1+1} > 0, \ldots, f_k > 0\} \cap P_i$$

and, moreover, this variety is a nonempty smooth compact hypersurface in $\overline{P}_i$. Besides,

$$\dim\bigl(cl(K_i' \setminus P_i) \cap K_i' \cap P_i\bigr) \le i - 2.$$

Proof. First we prove the inclusion

$$(K_i' \cap P_i) \supset \{g = \mu\} \cap \{f_{k_1+1} > 0, \ldots, f_k > 0\} \cap P_i.$$

We have

$$\{f_{k_1+1} > 0, \ldots, f_k > 0\} \cap P_i = \{f = 0, f_{k_1+1} > 0, \ldots, f_k > 0\} \cap P_i = W \cap P_i$$

since $\dim(W \cap P_i) = i$. By Lemma 5,

$$(\{f = 0, f_{k_1+1} > 0, \ldots, f_k > 0\} \cap P_i) \subset (K_i \cap P_i).$$

Intersecting both sides with the variety $\{g = \mu\}$, we obtain the $\supset$ inclusion.

To prove the inclusion $\subset$ observe that f_j is nonnegative everywhere on K_i for each $k_1 + 1 \le j \le k$ because $K_i \subset cl(W)$ (see Section 3).

On the other hand, f_j is nonzero everywhere on K_i' for $k_1 + 1 \le j \le k$ since $f_{k_1+1} \cdots f_k = \mu$. Thus, $K_i' \subset \{f_{k_1+1} > 0, \ldots, f_k > 0\}$ which proves $\subset$ inclusion.

Now let us prove that $K_i' \cap P_i$ is a nonempty smooth hypersurface in $\overline{P}_i$. Observe that $K_i' \cap P_i$ is bounded because P is compact, besides $K_i' \cap P_i$ is closed since its closure

$$K_i' \cap cl(P_i) = K_i' \cap \overline{P}_i \subset \{f_{k_1+1} > 0, \ldots, f_k > 0\} \cap \overline{P}_i = W \cap \overline{P}_i = W \cap P_i \subset P_i.$$

Since $\dim(\{f_{k_1+1} > 0, \ldots, f_k > 0\} \cap P_i) = i$, each connected component of the set $\{f_{k_1+1} > 0, \ldots, f_k > 0\} \cap P_i$ contains a connected component of the smooth hypersurface $\{g = \mu\} \cap \overline{P}_i$

(in $\overline{P}_i$) due to Morse theory (see [Hi 76]) and in view of (a). Moreover, each connected component of the hypersurface $\{g = \mu\} \cap \overline{P}_i$ either lies completely in the set $\{f_{k_1+1} > 0, \dots, f_k > 0\} \cap P_i$ or does not intersect this set.

Finally, the inequality

$$\dim\big(cl(K_i' \setminus P_i) \cap K_i' \cap P_i\big) \le i - 2$$

immediately follows from (b). The lemma is proved.

The next section is dedicated to the proof of the following lemma.

Lemma 10. *The number of i-faces P_i of P such that $K_i' \cap P_i$ is a nonempty compact smooth hypersurface in $\overline{P}_i$ and*

$$\dim\big(cl(K_i' \setminus P_i) \cap K_i' \cap P_i\big) \le i - 2,$$

does not exceed $(nkd)^{O(n)}$.

Theorem 2 immediately follows from Lemmas 9 and 10.

5. Extremal points of a linear function on K_i'

Take a linear form $L = \gamma_1 X_1 + \cdots + \gamma_n X_n$ with generic coefficients $\gamma_1, \dots, \gamma_n \in \mathbf{R}$. Fix P_i satisfying the conditions of Lemma 10 and denote by $L^{(P_i)}$ the restriction of L on $\overline{P}_i$. Then $L^{(P_i)}$ attains its maximal value, say $\theta_0^{(P_i)}$, on the compact set $K_i' \cap P_i$. Since $L^{(P_i)}$ is a generic linear form on $\overline{P}_i$ as well, the following two properties are fulfilled:

(i) $L^{(P_i)}$ attains the value $\theta_0^{(P_i)}$ at a unique point $v \in K_i' \cap P_i$ (cf. [Hi 76]);

(ii) the point v does not belong to $cl(K_i' \setminus P_i)$ (cf. the conditions of Lemma 10).

Indeed, the semialgebric set of linear forms on $\overline{P}_i$ for which the properties (i), (ii) fail, has the dimension less than the dimension of the set of all linear forms on $\overline{P}_i$, thus for the generic form L the properties (i), (ii) are valid.

Denote by V a connected component of $K_i' \cap P_i$ which contains v. The property (ii) implies that there exists $0 < r \in \mathbf{R}$ such that $B_v(r) \cap K_i' = B_v(r) \cap V$. Thus, L attains a local isolated maximum on K_i' at the point v by the property (i). Hence, there exists an element $0 < \zeta^{(P_i)} \in \mathbf{R}$ such that the values of L on the set $\partial B_v(r/2) \cap K_i'$ are less than $\theta_0^{(P_i)} - \zeta^{(P_i)}$.

Lemma 11. *The linear form L attains its maximal value $\theta^{(P_i)}$ on the set*

$$cl(\mathcal{K}_i \cap \{|g - \mu| < \varepsilon_1\}) \cap B_v(r/2)$$

(say, at a point w) and the values of L on the set

$$cl(\mathcal{K}_i \cap \{|g - \mu| < \varepsilon_1\}) \cap \partial B_v(r/2)$$

are less than $\mathrm{st}_0(\theta^{(P_i)}) - \zeta^{(P_i)}$. *Moreover,* $\mathrm{st}_0(\theta^{(P_i)}) = \theta_0^{(P_i)}$ *and* $\mathrm{st}_0(w) = v \in P_i$.

Proof. Due to Lemma 8 and the Remark after Lemma 3 from Section 2, we have:

$$\mathrm{st}_0 cl(\mathcal{K}_i \cap \{|g-\mu| < \varepsilon_1\}) = K_i'.$$

By the transfer principle, L attains its maximum $\theta^{(P_i)}$ on the closed bounded set $cl(\mathcal{K}_i \cap \{|g-\mu| < \varepsilon_1\}) \cap cl(B_v(r/2))$ at some point w. Then $\mathrm{st}_0(\theta^{(P_i)}) = \theta_0^{(P_i)}$ and $\mathrm{st}_0(w) = v$ (due to (i)).

Since

$$\mathrm{st}_0\big(cl(\mathcal{K}_i \cap \{|g-\mu| < \varepsilon_1\}) \cap \partial B_v(r/2)\big) \subset K_i' \cap \partial B_v(r/2)$$

we obtain the second statement of the lemma from the defining property of $\zeta^{(P_i)}$. Lemma is proved.

Lemma 11 states that L attains a local maximum on the set $cl(\mathcal{K}_i \cap \{|g-\mu| < \varepsilon_1\})$ at a point w such that $\mathrm{st}_0(w) \in P_i$. In order to estimate the number of such local maximum values of L we shall now construct a smooth hypersurface which is infinitely close to $cl(\mathcal{K}_i \cap \{|g-\mu| < \varepsilon_1\})$. After that it will be sufficient to bound the number of local maxima of L on this smooth hypersurface.

For a point y denote the coordinates of the gradient

$$\mathrm{grad}_y(f^{(y)} - \varepsilon_3) = (u_1, \dots, u_{n-i})$$

(cf. Definition). The set $\mathcal{K}_i \cap \{|g-\mu| < \varepsilon_1\}$ of the points $y = (y_1, \dots, y_n)$ can be represented as a union of $n-i$ semialgebraic sets of the form

$$U^{(i_0)} = \{f - \varepsilon_3 = 0, u_{i_0}^2 > 0, p_1 > 0, \dots, p_s > 0\} \subset \mathbf{R}_3^n, \quad 1 \le i_0 \le n-i$$

for some algebraic functions $p_1, \dots, p_s$ of the quadratic-irrational type introduced in Section 1, i.e., rational functions (with coefficients from $\mathbf{R}_2$) in $y_1, \dots, y_n$ and in

$$\sqrt{u_{i_0}^2},\ \sqrt{u_{i_0}^2 + u_{\pi_{i_0}(2)}^2}, \dots, \sqrt{u_{i_0}^2 + u_{\pi_{i_0}(2)}^2 + \cdots + u_{\pi_{i_0}(n-i)}^2} \tag{3}$$

(see Lemma 1). Here π_{i_0} is a permutation of $\{1, 2, \dots, n-i\}$ such that $\pi_{i_0}(1) = i_0$ (cf. Section 1).

Denote

$$q = \big(\varepsilon_5^2 - (f - \varepsilon_3)^2\big)(u_{i_0}^2 - \varepsilon_4)(p_1 - \varepsilon_4) \cdots (p_s - \varepsilon_4).$$

Introduce semialgebraic sets

$$\mathcal{U}_0^{(i_0)} = \{\varepsilon_5^2 > (f - \varepsilon_3)^2,\ u_{i_0}^2 > \varepsilon_4,\ p_1 > \varepsilon_4, \dots, p_s > \varepsilon_4\} \subset \mathbf{R}_5^n$$

and

$$\mathcal{U}^{(i_0)} = \{q = \varepsilon_6\} \cap \big(\mathcal{U}_0^{(i_0)}\big)^{(\mathbf{R}_6)} \subset \mathbf{R}_6^n.$$

Lemma 12 (cf. Lemmas 1, 4 in [GV 92]).

$$\mathrm{st}_3(\mathcal{U}^{(i_0)}) = cl(U^{(i_0)}).$$

Proof. Let us first show that to prove the lemma it is sufficient to establish the equality

$$\mathrm{st}_5(\mathcal{U}^{(i_0)}) = \partial \mathcal{U}_0^{(i_0)}. \tag{4}$$

Indeed, due to Lemma 3a),

$$\mathrm{st}_3(\partial \mathcal{U}_0^{(i_0)}) \subset \mathrm{st}_3(cl(\mathcal{U}_0^{(i_0)})) = cl(U^{(i_0)}),$$

thus, due to (4), $\mathrm{st}_3(\mathcal{U}^{(i_0)}) \subset cl(U^{(i_0)})$.

On the other hand, $cl(U^{(i_0)}) = \partial(U^{(i_0)})$ because $U^{(i_0)} \subset \{f = \varepsilon_3\}$ and thereby $U^{(i_0)}$ contains no internal points. Hence, Lemma 3b) implies that $cl(U^{(i_0)}) \subset \mathrm{st}_3(\partial \mathcal{U}_0^{(i_0)})$. It follows from (4) that $cl(U^{(i_0)}) \subset \mathrm{st}_3(\mathcal{U}^{(i_0)})$. This would prove the lemma, provided that (4) holds.

Now we prove (4) starting with the inclusion $\subset$.

Let a point $y \in \mathcal{U}^{(i_0)}$ and the standard part $x = \mathrm{st}_5(y)$ be definable. Then $q(x) = 0$ and

$$\varepsilon_5^2 \geq (f - \varepsilon_3)^2, u_{i_0}^2 \geq \varepsilon_4, p_1 \geq \varepsilon_4, \ldots, p_s \geq \varepsilon_4.$$

Suppose that $x \notin \partial \mathcal{U}_0^{(i_0)}$. Therefore there exists $0 < r_0 \in \mathbf{R}_5$ such that either $B_x(r_0) \subset \mathcal{U}_0^{(i_0)}$ or $B_x(r_0) \cap \mathcal{U}_0^{(i_0)} = \emptyset$.

If $B_x(r_0) \subset \mathcal{U}_0^{(i_0)}$ we get a contradiction with $q(x) = 0$.

If $B_x(r_0) \cap \mathcal{U}_0^{(i_0)} = \emptyset$ we conclude that the intersection $B_x(r_0) \cap (\mathcal{U}_0^{(i_0)})^{(\mathbf{R}_6)} = \emptyset$. Since y belongs to this intersection we again get a contradiction which proves tie inclusion $\subset$ in (4).

To prove the inclusion $\supset$ in (4) take a point $x \in \partial \mathcal{U}_0^{(i_0)}$. Observe that q is positive everywhere on $\mathcal{U}_0^{(i_0)}$ and q vanishes everywhere on $\partial \mathcal{U}_0^{(i_0)}$, in particular $q(x) = 0$.

Suppose that $x \notin \mathrm{st}_5(\mathcal{U}^{(i_0)})$. Then there exists $0 < r_1 \in \mathbf{R}_5$ such that $B_x(r_1) \cap \mathcal{U}^{(i_0)} = \emptyset$ (cf. the proof of Lemma 2). Consider the decomposition of the intersection

$$B_x(r_1) \cap \mathcal{U}_0^{(i_0)} = \bigcup_j \mathcal{U}_j$$

into its connected components (which are also semialgebraic sets and there is a finite number of them, see e.g. [GV 88]). Since $x \in cl(B_x(r_1) \cap \mathcal{U}_0^{(i_0)})$ there is j_0 for which $x \in cl(\mathcal{U}_{j_0})$. All the values of the polynomial q on $\mathcal{U}_{j_0}$ form a connected semialgebraic subset $\Xi \subset \mathbf{R}_5$. So, Ξ is an interval (or a point) with endpoints ξ_1, $\xi_2 \in \mathbf{R}_5$, $\xi_1 \leq \xi_2$, the set Ξ could be either closed, either open, or semi-open. Observe that $\xi_1 \geq 0$ because $\mathcal{U}_{j_0} \subset \mathcal{U}_0^{(i_0)}$. On the other hand, $\xi_1 = 0$ since $q(x) = 0$ and $x \in cl(\mathcal{U}_{j_0})$. Obviously, $\xi_2 > 0$. Due to the transfer principle, q attains on the set $\mathcal{U}_{j_0}^{(\mathbf{R}_6)}$ all the values from the interval $(0, \xi_2)^{(\mathbf{R}_6)}$. In particular, there exists a point $y \in \mathcal{U}_{j_0}^{(\mathbf{R}_6)}$ such that $q(y) = \varepsilon_6$. Then $y \in B_x(r_1) \cap \mathcal{U}^{(i_0)}$. The obtained contradiction completes the proof of the lemma.

Lemma 13. *For a certain $1 \le i_0 \le n - i$ the linear form L attains its maximal value $\theta_1^{(P_i)}$ on the set $cl(\mathcal{U}^{(i_0)}) \cap B_v(r/2)$ at a certain point w_1, and the values of L on the set $cl(\mathcal{U}^{(i_0)}) \cap \partial B_v(r/2)$ are less than $\mathrm{st}_0(\theta_1^{(P_i)}) - \zeta^{(P_i)}$. Moreover, $\mathrm{st}_3(\theta_1^{(P_i)}) = \theta^{(P_i)}$ and $\mathrm{st}_0(w_1) = v \in P_i$.*

Proof. Since

$$\mathcal{K}_i \cap \{|g - \mu| < \varepsilon_1\} = \bigcup_{1 \le i_0 \le n-i} U^{(i_0)}$$

there is $1 \le i_0 \le n - i$ such that $w \in cl(U^{(i_0)})$ (see Lemma 11). The linear form L attains its maximum $\theta_1^{(P_i)}$ on the bounded closed set $cl(\mathcal{U}^{(i_0)}) \cap cl(B_v(r/2))$ at a point w_1. Using Lemma 12 and the equaliy $\mathrm{st}_3(\mathcal{U}^{(i_0)}) = \mathrm{st}_3(cl(\mathcal{U}^{(i_0)}))$ (see the Remark in Section 2), we get: $\mathrm{st}_3(\theta_1^{(P_i)}) = \theta^{(P_i)}$. Due to (i), $\mathrm{st}_0(w_1) = v$.

The values of L on the set $cl(\mathcal{U}^{(i_0)}) \cap \partial B_v(r/2)$ are less than $\theta_0^{(P_i)} - \zeta^{(P_i)}$ due to the similar statement in Lemma 11, taking into the account that

$$\mathrm{st}_3(cl(\mathcal{U}^{(i_0)}) \cap \partial B_v(r/2)) \subset \partial B_v(r/2).$$

Lemma is proved.

Since w_1 is a local maximum of L on the set $cl(\mathcal{U}^{(i_0)})$ we obtain the following corollary.

Corollary. *The number of i-faces P_i satisfying the conditions of Lemma 10 does not exceed the number of all the values of local maxima of the linear form L on the set*

$$\bigcup_{1 \le i_0 \le n-i} cl(\mathcal{U}^{(i_0)}).$$

Lemma 14. *$\mathcal{U}^{(i_0)}$ is a smooth closed hypersurface.*

Proof. First we prove that $cl(\mathcal{U}^{(i_0)}) = \mathcal{U}^{(i_0)}$. Let a point $x \in cl(\mathcal{U}^{(i_0)})$. Then

$$q(x) = \varepsilon_6, \varepsilon_5^2 \ge (f(x) - \varepsilon_3)^2, u_{i_0}^2(x) \ge \varepsilon_4, p_1(x) \ge \varepsilon_4, \dots, p_s(x) \ge \varepsilon_4.$$

Neither among the latter inequalities could convert into an equality because $q(x) = \varepsilon_6 \ne 0$, hence $x \in \mathcal{U}^{(i_0)}$.

Observe that in the open semialgebraic set $\{u_{i_0}^2 > 0\}$ all the square roots (3) are positive. Therefore all algebraic functions $p_1, \dots, p_s$ occuring in $U^{(i_0)}$ are smooth, hence q is smooth as well. Because of Lemma 4, ε_6 is not a critical value of q in the set $\{u_{i_0}^2 > 0\}$. Then the implicit function theorem implies the lemma.

Finally, let us prove the following lemma.

Lemma 15. *The number of local maxima of L on $\mathcal{U}^{(i_0)}$ does not exceed $(nkd)^{O(n)}$.*

Proof. Because of Lemma 14, the number of local maxima of L on $\mathcal{U}^{(i_0)}$ does not exceed the number of connected components of the semialgebraic set

$$M = \{0 = q - \varepsilon_6 = \gamma_i \frac{\partial q}{\partial X_j} - \gamma_j \frac{\partial q}{\partial X_i},\ 1 \le i < j \le n\} \subset \mathbf{R}_6^n$$

(by the Lagrange multiplier theorem, see, e.g., Ch. 4 in [Th 77] and taking into the account the transfer principle).

Replace each occurrence of the square root

$$\sqrt{u_{i_0}^2 + u_{\pi_{i_0}(2)}^2 + \cdots + u_{\pi_{i_0}(m)}^2},$$

$1 \le m \le n - i$ in q by a new variable Z_m. Denote the resulting rational function by $Q \in \mathbf{R}_5[X_1, \ldots, X_n](Z_1, \ldots, Z_{n-i})$ (cf. Section 1).

Introduce the semialgebraic set

$$\mathcal{M} = \{0 = Q - \varepsilon_6 = \gamma_i \frac{\partial Q}{\partial X_j} - \gamma_j \frac{\partial Q}{\partial X_i},\ 1 \le i < j \le n,$$

$$Z_m > 0,\ Z_m^2 = u_{i_0}^2 + u_{\pi_{i_0}(2)}^2 + \cdots + u_{\pi_{i_0}(m)}^2,\ 1 \le m \le n - i\} \subset \mathbf{R}_6^{2n-i}.$$

Consider the linear projection

$$\rho : \mathbf{R}_6^{2n-i} \longrightarrow \mathbf{R}_6^n, \quad \rho(X_1, \ldots, X_n, Z_1, \ldots, Z_{n-i}) = (X_1, \ldots, X_n).$$

Then $\rho(\mathcal{M}) = M$. Hence the number of connected components of M is less or equal to the number of connected components of $\mathcal{M}$.

Observe that the degrees of rational functions occuring in $\mathcal{M}$ can be bounded from above by $(knd)^{O(1)}$ due to Lemma 1 and Definition. Therefore, the number of connected components of $\mathcal{M}$ does not exceed $(knd)^{O(n)}$ by [Mi 64].

This completes the proof of the lemma.

Lemma 15 together with Corollary to Lemma 13 imply Lemma 10 and thereby Theorems 2 and 1.

6. Lower bounds for concrete polyhedra

In this section we give an application of the lower bound from Theorem 1 to a concrete class of polyhedra. We follow the construction of cyclic polyhedra (see [MS 71]), used in the analysis of the simplex method.

Take any $m > \Omega(n^2)$ points in $\mathbf{R}^n$ of the form $(t_j, t_j^2, \ldots, t_j^n)$ for pairwise distinct t_j, $1 \le j \le m$. Consider the convex hull of these points and denote by $P_{n,m} \subset \mathbf{R}^n$ its dual polyhedron [MS 71]. Then $P_{n,m}$ has m faces of the highest dimension $n - 1$ and the number of faces of all dimensions

$$N > \binom{m - \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} > m^{\Omega(n)}$$

(see [MS 71]).

Therefore, Theorem 1 implies that the complexity of testing membership to $P_{n,m}$ is bounded by $\Omega(\log N) > \Omega(n \log m)$.

We would like to mention that Section 4 of [GKV 94] provides a weaker bound $\Omega(\log m)$ even for algebraic *computation* trees.

7. Open problems

1. Is it possible to get rid of any lower bound assumption on N in Theorem 1?
2. Is it possible to extend the result of Theorem 1 to algebraic computation trees?

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