On Primitive Cellular Algebras

Sergei Evdokimov * **Ilia Ponomarenko** †

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Abstract

First we define and study the exponentiation of a cellular algebra by a permutation group which is analogous to the corresponding operation (the wreath product in primitive action) in permutation group theory. Necessary and sufficient conditions for the resulting cellular algebra to be primitive and Schurian are given. This enables us to construct infinite series of primitive non-Schurian algebras. Also we define and study for cellular algebras the notion of a base which is similar to that for permutation groups. We present an upper bound for the size of an irredundant base of a primitive cellular algebra in terms of the parameters of its standard representation. This produces new upper bounds for the order of the automorphism group of such an algebra and in particular for the order of a primitive permutation group. Finally we generalize to 2-closed primitive algebras some classical theorems for primitive groups and show that the hypothesis for a primitive algebra to be 2-closed is essential.

*St.Petersburg Institute for Informatics and Automation of the Academy of Sciences of Russia, 199178 St.Petersburg and University of Bonn, 53117 Bonn. E-mail: evdokim@pdmi.ras.ru. Research supported by the Volkswagen-Stiftung Program on Computational Complexity.

†St.Petersburg Department of Mathematical Institute of the Academy of Sciences of Russia, 191011 St.Petersburg and University of Bonn, 53117 Bonn. E-mail: ponom-ko@pdmi.ras.ru. Research supported by the Volkswagen-Stiftung Program on Computational Complexity.

1 Introduction

There are at least three mathematical environments carring in a natural way into cellular (coherent) algebras introduced by Weisfeiler and Lehman and independently by Higman (see [11], [8]). These are by definition matrix algebras over $\mathbf{C}$ closed under the Hadamard multiplication and the Hermitian conjugation and containing the identity matrix and the all one matrix. The first environment is permutation group theory where a cellular algebra arises as the centralizer algebra $\mathcal{Z}(G)$ of a permutation group G (see chapter 5 of [13]). Another field is algebraic combinatorics, especially the association scheme and design theories. All objects in this context can be considered as special cases of coherent configurations and cellular algebras arise as the adjacency algebras of them (see [8]). Finally, there is a cellular algebra approach to the Graph Isomorphism Problem which is one of the most well-known unsolved problems in computational complexity theory. The cornerstone of this approach consists in associating to a graph the smallest cellular algebra containing its adjacency matrix. There is a canonical polynomial-time procedure for this which reduces the Graph Isomorphism Problem to the corresponding problem for cellular algebras (see [11]).

One of the advantages in studying cellular algebras consists in the following observation: on one hand the axioms defining them are less restrictive than those of groups and on the other hand they are not so amorphic objects as graphs. In other words, cellular algebras accumulate algebraic features of groups and combinatorial features of graphs. To demonstrate this we briefly discuss below three topics concerning them and playing the central role in this paper: representations, Schurity and primitivity.

Let W be a cellular algebra on a finite set V, i.e. a cellular subalgebra of the full matrix algebra Mat_V (the set of all complex matrices whose rows and columns are indexed by the elements of V). Then from the representation theory point of view W can be viewed as an algebra equipped with a faithful linear representation

$$\rho : W \to \mathrm{Mat}_V .$$

Since W is a semisimple algebra over $\mathbf{C}$, the representation ρ is completely reducible. One of our goals here is to present a connection between the structure properties of W and the representation parameters of ρ such as its irreducible representation multiplicities and degrees. It should be mentioned that our approach is different from that of [2] where commutative cellular algebras arising from distance regular graphs have been studied.

Going over to the following topic let us define the automorphism group $\mathrm{Aut}(W)$ of a cellular algebra W to be the subgroup of the symmetric group $\mathrm{Sym}(V)$ consisting of all permutations the permutation matrices of which commute with all matrices of W. At the beginning of the development of cellular algebra theory there was a conjecture that each cellular algebra W is Schurian (see [7] for the explanation of the term), i.e. that the following equality holds:

$$W = \mathcal{Z}(\mathrm{Aut}(W)). \tag{1}$$

If it was so, then we would have a complete characterization of distance regular graphs (see [2]) and a polynomial-time algorithm for the Graph Isomorphism Problem (see [11]). However this is not the case and some examples of non-Schurian cellular algebras can be found in [13] and [7]. This leads to the following problem (Schurity problem): determine whether a given cellular algebra is Schurian. The Schurity problem is the second topic of our paper.

The combinatorial features of cellular algebras are based on the fact that the last are closed with respect to the Hadamard multiplication. This implies that each cellular algebra W contains a uniquely determined linear base consisting of {0,1}-matrices summing up to J_V (the all one matrix of Mat_V), which enables us to view it as the adjacency algebra of a coherent configuration. Using this base we can define homogeneous and primitive cellular algebras corresponding in the sense explained below to transitive and primitive permutation groups. Namely, we call W homogeneous if it contains exactly one diagonal {0,1}-matrix (the identity matrix I_V), and primitive if it contains exactly two {0,1}-matrices (I_V and J_V) coinciding with the adjacency matrix of an equivalence relation on V. These definitions imply that a permutation group G is transitive (resp. primitive) if and only if its centralizer algebra $\mathcal{Z}(G)$ is homogeneous (resp. primitive) as a cellular algebra. Like primitive groups in permutation group theory primitive cellular algebras are "building blocks" for arbitrary cellular algebras. The study of them is the third topic of the paper.

The main results of the paper are contained in sections 3, 4 and 5 (section 2 presents exact definitions and notation concerning cellular algebras).

In section 3 we define a new operation, the exponentiation of a cellular algebra W by a permutation group K, the result of which is a cellular algebra $W \uparrow K$. It is similar to the corresponding operation for permutation groups (see [7]). It is known that the exponentiation $G \uparrow K$ of permutation groups G and K is primitive iff G is primitive and nonregular and K is transitive. We generalize this result by showing (theorem 3.4) that the algebra $W \uparrow K$ is primitive iff W is primitive and nonregular and K is transitive. We also show that given an arbitrary permutation group K, the cellular algebra $W \uparrow K$ is Schurian iff so is W (theorem 3.3). These theorems enable us to construct infinite series of non-Schurian primitive algebras (see the end of subsection 3.3).

In section 4 we define for cellular algebras the notion of a base which is similar to that for permutation groups. It is closely related to the Schurity problem: according to [4] an m-closed cellular algebra having a base of size $m-1$ is Schurian (as to the discussion of higher closed algebras see below). The main result of this section (theorem 4.3) shows that the size of each irredundant base of a primitive cellular algebra W does not exceed any ratio m_P/n_P where m_P (resp. n_P) is the multiplicity (resp. degree) of a nonprincipal primitive central idempotent P of W. Using this upper bound (see also theorem 4.10) we present two upper bounds for the order of $\mathrm{Aut}(W)$ in terms of the relation degrees of W and the above ratios (corollary 4.12). Note that each of these results gives rise to the corresponding result for primitive permutation groups.

In section 5 we continue the investigation of the m-closure of a cellular algebra introduced in [4] (the exact definition of m-closure can also be found in this section). It can be considered as an approximation of a cellular algebra W on V to the Schurian closure $\mathrm{Sch}(W) = \mathcal{Z}(\mathrm{Aut}(W))$ of it (see [4]):

$$W = W^{(1)} \leq \ldots \leq W^{(n)} = \mathrm{Sch}(W)$$

where $W^{(m)}$ is the m-closure of W and n is the cardinality of V. Assuming a primitive cellular algebra to be 2-closed (i.e. coinciding with its 2-closure) we generalize to it some classical theorems holding for a primitive permutation group. For example, we characterize 2-closed primitive algebras W as homogeneous ones for which any algebra W_v, $v \in V$ (the analog of a one-point stabilizer in permutation group case) is a minimal overalgebra of W (theorem 5.5). In particular, this implies that such a W equals $W_u \cap W_v$ for any two different points u and v of V unless it is regular of prime degree (theorem 5.6). We also generalize to these algebras a well-known theorem saying that each 3/2-transitive group is either primitive or a Frobenius group (theorem 5.9). Finally, we give an example showing that all the theorems are not true if the hypothesis for a primitive algebra to be 2-closed is omitted (see the end of subsection 5.3).

Notation. As usual by $\mathbf{C}$ we denote the complex field.

Throughout the paper V denotes a finite set with $n = |V|$ elements. By relations on V me mean subsets of $V \times V$. For a relation R on V we set $R^T = \{(u,v) : (v,u) \in R\}$ and $R(v) = \{u : (v,u) \in R\}$ where $v \in V$. If E is an equivalence (i.e. reflexive, symmetric and transitive relation) on V, then V/E denotes the set of all classes modulo E.

The algebra of all complex matrices whose rows and columns are indexed by the elements of V is denoted by Mat_V, its unity element (the identity matrix) by I_V and the all one matrix by J_V. For $U \subset V$ the algebra Mat_U is in a natural way identified with a subalgebra of Mat_V.

For $U, U' \subset V$ we denote by $J_{U,U'}$ the {0,1}-matrix with 1's exactly at the places belonging to $U \times U'$. If $A \in \mathrm{Mat}_V$, then A^T denotes the transpose and A^* the Hermitian conjugate matrix.

If $\varphi : V \to V'$ is a bijection, then A^φ denotes the image of a matrix A with respect to the natural algebra isomorphism from Mat_V to $\mathrm{Mat}_{V'}$ induced by φ.

The group of all permutations of V is denoted by $\mathrm{Sym}(V)$.

For integers l, m with $l \le m$ by $[l, m]$ we denote the set $\{l, l+1, \ldots, m\}$.

2 Cellular algebras

All undefined below terms concerning cellular algebras and permutation groups can be found in [11] and [13] respectively.

2.1 By a *cellular algebra* on V we mean a subalgebra W of Mat_V for which the following conditions are satisfied:

(C1) $I_V, J_V \in W$;

(C2) $\forall A \in W : \quad A^* \in W$;

(C3) $\forall A, B \in W : \quad A \circ B \in W$,

where $A \circ B$ is the Hadamard (componentwise) product of the matrices A and B. It follows from (C2) that W is a semisimple algebra over $\mathbf{C}$.

Each cellular algebra W has a uniquely determined linear basis $\mathcal{R} = \mathcal{R}(W)$ (the *standard basis* of W) consisting of {0,1}-matrices such that

$$\sum_{R \in \mathcal{R}} R = J_V \quad \text{and} \quad R \in \mathcal{R} \Leftrightarrow R^* \in \mathcal{R}. \tag{2}$$

Set $\mathrm{Cel}(W) = \{U \subset V : \ I_U \in \mathcal{R}\}$ and $\mathrm{Cel}^*(W) = \{U \subset V : \ I_U \in W\}$. Each element of $\mathrm{Cel}(W)$ is called a *cell* of W. It is easy to see that

$$V = \bigcup_{U \in \mathrm{Cel}(W)} U \qquad \text{(disjoint union)}.$$

The algebra W is called *homogeneous* if $|\,\mathrm{Cel}(W)| = 1$.

For $U, U' \in \mathrm{Cel}^*(W)$ set $\mathcal{R}_{U,U'} = \{R \in \mathcal{R} : \ R \circ J_{U,U'} = R\}$. Then

$$\mathcal{R} = \bigcup_{U,U' \in \mathrm{Cel}(W)} \mathcal{R}_{U,U'} \qquad \text{(disjoint union)}.$$

Moreover, for two cells U, U' the number of 1's in the uth row (resp. vth column) of the matrix $R \in \mathcal{R}_{U,U'}$ does not depend on the choice of $u \in U$ (resp. $v \in U'$). This number is denoted by $d_{out}(R)$ (resp. $d_{in}(R)$). If W is homogeneous, then $d_{out}(R) = d_{in}(R)$ for all $R \in \mathcal{R}$ and we use the notation $d(R)$ for this number and call it the *degree* of R. In this case we have

$$\sum_{R \in \mathcal{R}} d(R) = |V|. \tag{3}$$

For each cell $U \in \mathrm{Cel}(W)$ we view the subalgebra $I_U W I_U$ of W as a cellular algebra on U. It is denoted by W_U and called the *homogeneous component* of W corresponding to U. The basis matrices of W_U are in 1-1 correspondence to the matrices of $\mathcal{R}_{U,U}$.

Each matrix $R \in \mathcal{R}$ being a $\{0,1\}$-matrix is the adjacency matrix of some relation on V called a *basis relation* of W. By (2) the set of all of them form a partition of $V \times V$ which can be interpreted as a coherent configuration on V (see [8]). We often use for this set the same notation $\mathcal{R}$ and save for basis relations all the notations introduced for basis matrices.

2.2 A large class of cellular algebras comes from permutation groups as follows (see [11]). Let $G \leq \mathrm{Sym}(V)$ be a permutation group and

$$\mathcal{Z}(G) = \{A \in \mathrm{Mat}_V : \ A^g = A, \ g \in G\}$$

be its centralizer algebra. Then $\mathcal{Z}(G)$ is a cellular algebra on V. Its basis relations are exactly the 2-orbits of G. In particular, $\mathrm{Cel}(\mathcal{Z}(G)) = \mathrm{Orb}(G)$ where $\mathrm{Orb}(G)$ is the set of G-orbits.

A cellular algebra W is called *semiregular* if $d_{in}(R) = d_{out}(R) = 1$ for all $R \in \mathcal{R}(W)$. A homogeneous semiregular algebra is called *regular*. It is easy to see that semiregular (regular) algebras are exactly the centralizer algebras of semiregular (regular) permutation groups.

Two cellular algebras W and W' on V and V' are called *isomorphic* ($W \cong W'$) if $W^\varphi = W'$ (as sets) for some bijection $\varphi : V \to V'$ called an *isomorphism* from W to W'. Clearly, φ induces a bijection between the sets $\mathcal{R}(W)$ and $\mathcal{R}(W')$. The group of all isomorphisms from W to itself contains a normal subgroup

$$\mathrm{Aut}(W) = \{\varphi \in \mathrm{Sym}(V) : \ A^\varphi = A, \ A \in W\}$$

called the *automorphism group* of W. If $W = \mathcal{Z}(\mathrm{Aut}(W))$, then W is called *Schurian*. It is easy to see that W is Schurian iff the set of its basis relations coincides with the set of

2-orbits of $\mathrm{Aut}(W)$. We note that semiregular cellular algebras are Schurian. It follows from [13] that there exist cellular algebras which are not Schurian (see also [7]). The smallest example of a non-Schurian cellular algebra which we know is the commutative algebra $T(15)$ on 15 points corresponding to the skew Hadamard matrix of order 16 (see [7] for the explicit description). This algebra has three basis relations whose degrees are 1, 7, 7 and is primitive in the sense of subsection 2.3.

Along with the notion of an isomorphism we consider for cellular algebras that of a weak one. Namely, cellular algebras W on V and W' on V' are called *weakly isomorphic* if there exists an algebra isomorphism $f : W \to W'$ such that $f(A^*) = f(A)^*$ and $f(A \circ B) = f(A) \circ f(B)$ for all $A, B \in W$. In this case $|V| = |V'|$, $f(\mathcal{R}(W)) = \mathcal{R}(W')$ and f induces a bijection from $\mathrm{Cel}(W)$ on $\mathrm{Cel}(W')$. Any such f is called a *weak isomorphism* from W to W'. Note that each isomorphism from W to W' induces in a natural way a weak isomorphism between these algebras.

2.3 Let W be a cellular algebra on V and E be an equivalence on V. We say that E is an *equivalence* of W if it is the union of basis relations of W. The equivalences of W with the adjacency matrices I_V and J_V are called *trivial*. Suppose now that W is homogeneous. We call W *imprimitive* if it has a nontrivial equivalence. Otherwise it is called *primitive* unless $|V| = 1$. We stress that a cellular algebra on a one-point set is neither imprimitive nor primitive according to this definition.

Lemma 2.1 ([11]) *Let W be a primitive cellular algebra having a nonreflexive basis relation of degree 1. Then $W \cong \mathcal{Z}(Z_p)$ where Z_p is a regular permutation group on p points, p being a prime.*$\blacksquare$

If W is primitive, then according to [11] each nonreflexive basis relation of W is strongly connected (in the corresponding graph any two vertices are connected by a directed path). In other words, given a basis matrix R of W, $R \neq I_V$, each basis matrix S of W enters R^{i_S} for some positive integer i_S, i.e. $S \circ R^{i_S} \neq 0$. If W is not regular, then i_S can be chosen the same for all S by lemma 2.1.

2.4 The set of all cellular algebras on V is put in order by inclusion. The greatest and the least elements of the set are respectively the full matrix algebra Mat_V and the simplex $S(V) = \mathcal{Z}(\mathrm{Sym}(V))$, i.e. the algebra with the linear base $\{I_V, J_V\}$. For cellular algebras W and W' we write $W \leq W'$ if W is a subalgebra of W'.

Given a subset X of Mat_V, we denote by $[X]$ the cellular closure of X, i.e. the smallest cellular algebra containing X. If W is a cellular algebra on V, then $W[X]$ denotes $[W \cup X]$. If $X = \{I_{\{u\}} : \ u \in U\}$ where U is a subset of V, we use notation $W_{[U]}$ instead of $W[X]$ and set $W_{v_1,\dots,v_s} = W_{[\{v_1,\dots,v_s\}]}$ for $v_1, \dots, v_s \in V$.

3 The exponentiation of cellular algebras

3.1 Let $W \leq \mathrm{Mat}_V$ be a cellular algebra with $\mathcal{R}$ as the standard basis and Φ be a group of weak isomorphisms from W to itself. For $R \in \mathcal{R}$ we set

$$R^\Phi = \frac{1}{|\Phi_R|} \sum_{\varphi \in \Phi} R^\varphi$$

where $\Phi_R = \{\varphi \in \Phi : \ R^\varphi = R\}$ (we use notation R^φ instead of $\varphi(R)$). Then clearly R^Φ is a $\{0,1\}$-matrix and two such matrices either coincide or orthogonal with respect to the Hadamard multiplication. It is easy to see that

$$\sum_{A \in \mathcal{R}^\Phi} A = J_V \quad \text{and} \quad A^T \in \mathcal{R}^\Phi \ \Leftrightarrow \ A \in \mathcal{R}^\Phi \tag{4}$$

where $\mathcal{R}^\Phi = \{R^\Phi : \ R \in \mathcal{R}\}$. Set $W^\Phi = [\mathcal{R}^\Phi] \leq \mathrm{Mat}_V$.

Lemma 3.1 *The set $\mathcal{R}^\Phi$ is the standard basis of the cellular algebra W^Φ.*

Proof. To prove the statement it suffices by (4) to verify that the product of two matrices belonging to $\mathcal{R}^\Phi$ is a linear combination of matrices from $\mathcal{R}^\Phi$. Let us denote by $c_{R,S}^T$ the structure constants of the algebra W with respect to $\mathcal{R}$. Then

$$R^\Phi S^\Phi = \frac{1}{m} \sum_{\varphi,\psi \in \Phi} R^\varphi S^\psi = \frac{1}{m} \sum_{\varphi,\psi' \in \Phi} \left(R S^{\psi'}\right)^\varphi = \frac{1}{m} \sum_{\varphi,\psi' \in \Phi} \sum_{T \in \mathcal{R}} c_{R,S^{\psi'}}^T T^\varphi =$$

$$= \frac{1}{m} \sum_{\psi' \in \Phi} \sum_{T \in \mathcal{R}} c_{R,S^{\psi'}}^T \Big(\sum_{\varphi \in \Phi} T^\varphi\Big) = \frac{1}{m} \sum_{\psi' \in \Phi} \sum_{T \in \mathcal{R}} c_{R,S^{\psi'}}^T |\Phi_T| T^\Phi$$

where $m = |\Phi_R||\Phi_S|$. This proves the required statement.■

Let $W \leq \mathrm{Mat}_V$ be a cellular algebra and let K be a permutation group on a set X. Defining the action of the group K on V^X by

$$(\{v_x\}_{x \in X})^k = \{v_{x^{k^{-1}}}\}_{x \in X}, \quad v_x \in V, k \in K$$

we can view K as a subgroup of the group of weak isomorphisms of W^X to itself where $W^X = W \otimes \cdots \otimes W$ (X times). By lemma 3.1 this defines a cellular algebra on V^X denoted by $W \uparrow K$ and called the *exponentiation* of W by K.

The following properties of the exponentiation are straightforward from the definition.

Proposition 3.2 *The following canonical isomorphisms take place:*

(1) $W \uparrow (K_1 \oplus K_2) \cong (W \uparrow K_1) \otimes (W \uparrow K_2)$;

(2) $W \uparrow (K_1 \wr K_2) \cong (W \uparrow K_1) \uparrow K_2$;

(3) $(W_1 \otimes W_2) \uparrow K \cong (W_1 \uparrow K) \otimes (W_2 \uparrow K)$

where W, W_1, W_2 are cellular algebras, K_1, K_2, K are permutation groups and $\oplus$ (resp. $\wr$) denotes the direct sum (resp. the wreath product in imprimitive action) of permutation groups.■

3.2 The above operation is closely related to the exponentiation of permutation groups called also the wreath product in primitive action (see for instance [7]). The exponentiation $G \uparrow K$ of a permutation group $G \leq \mathrm{Sym}(V)$ by a permutation group $K \leq \mathrm{Sym}(X)$ is by definition the group consisting of permutations $(\{g_x\}_{x \in X}, k)$ with $g_x \in G, k \in K$ acting on the set V^X by

$$\{v_x\}^{(\{g_x\},k)} = \{v_{x^{k-1}}^{g_{x^{k-1}}}\}.$$

The following inclusions are valid

$$\mathrm{Aut}(W)\uparrow K \leq \mathrm{Aut}(W\uparrow K) \leq \mathrm{Aut}(W)\uparrow K^{(1)} \tag{5}$$

where $K^{(1)}$ is the 1-closure of K, i.e. the product of the symmetric groups acting on the orbits of K. The first inclusion is straightforward. To check the second one we note that $\mathrm{Aut}(S(V)\uparrow \mathrm{Sym}(X)) = \mathrm{Sym}(V)\uparrow \mathrm{Sym}(X)$ (the both groups clearly have the same 2-orbits and the second one is 2-closed, see [7]). Then each permutation from $\mathrm{Aut}(W\uparrow K)$ is of the form $\sigma = (\{g_x\}, k)$, $g_x \in \mathrm{Sym}(V)$, $k \in K$. Moreover, $g_x \in \mathrm{Aut}(W)$ and $k \in K^{(1)}$, which follows from considering the action of σ on the basis relations of $W\uparrow K$ having the form R^X and $R^O \times S^{X\setminus O}$ where R, S are different basis relations of W and O is an orbit of K.

The Schurity problem for the exponentiation is solved by the following statement.

Theorem 3.3 *The cellular algebra $W\uparrow K$ is Schurian iff so is W.*

Proof. Let $K \leq \mathrm{Sym}(X)$. If W is Schurian, then clearly so is W^X. This implies by the definition of the exponentiation, that the basis relations of $W\uparrow K$ are of the form $O^K = \cup_{k\in K} O^k$ where O is a 2-orbit of $\mathrm{Aut}(W^X)$. Since $\mathrm{Aut}(W)\uparrow K$ acts transitively on O^K, the sufficiency follows from the left side inclusion of (5).

Conversely, let R be an arbitrary basis relation of W. Then the Schurity of $W\uparrow K$ implies that the group $\mathrm{Aut}(W\uparrow K)$ acts transitively on the basis relation R^X of $W\uparrow K$. On the other hand, by the right side inclusion of (5) each permutation from $\mathrm{Aut}(W\uparrow K)$ is of the form $(\{g_x\}, k)$ with $g_x \in \mathrm{Aut}(W)$ for all $x \in X$. So the group $\prod_{x\in X}\mathrm{Aut}(W)$ acts transitively on the set of pairs $(\{u\}_{x\in X}, \{v\}_{x\in X})$ where $(u, v) \in R$. Thus R is a 2-orbit of $\mathrm{Aut}(W)$ and so W is Schurian.■

3.3 If W is homogeneous, then $W\uparrow K$ is a cellular subalgebra of the homogeneous algebra W^X. So it is homogeneous too. On the other hand, if U is a cell of W, then clearly $U^X \in \mathrm{Cel}(W\uparrow K)$. Thus the algebra $W\uparrow K$ is homogeneous if and only if so is W. The following statement characterizes the case of primitive exponentiation.

Theorem 3.4 *Let $W \leq \mathrm{Mat}_V$ be a cellular algebra and $K \leq \mathrm{Sym}(X)$ be a permutation group. Then $W\uparrow K$ is primitive iff K is transitive and W is primitive and nonregular.*

Proof. Let $W\uparrow K$ be a primitive cellular algebra. Then $|V| > 1$. If $O \in \mathrm{Orb}(K)$, then the equivalence on V^X the classes of which are defined by the equality of coordinates outside of O is a nontrivial equivalence of $W\uparrow K$. So K is a transitive group. Analogously, an equivalence E of W produces the equivalence of $W\uparrow K$ the classes of which are $\prod_{x\in X} U_x$, $U_x \in V/E$. This shows that W is primitive. Finally, W can not be a regular algebra for otherwise the relation with the adjacency matrix $\sum_{R\in\mathcal{R}} R^X$ where $\mathcal{R} = \mathcal{R}(W)$ would be a nontrivial equivalence of $W\uparrow K$.

Conversely, let us consider the adjacency matrix E of some equivalence of $W\uparrow K$. Suppose that $E \neq I_{V^X}$ and set

$$\mathcal{S} = \{R \in \mathcal{R}(W^X):\ R\circ E = R\}.$$

Then each matrix $R \in \mathcal{S}$ can be written in the form $R = \otimes_{x\in X} R_x$ where $R_x \in \mathcal{R}$ for all x. Let us choose a matrix R from $\mathcal{S}$ such that the number $n_R = |\{x \in X:\ R_x = I_V\}|$ be as small as possible. It follows from the choice of E that $R \neq I_{V^X}$ and so $n_R < |X|$.

We will show that $n_R = 0$. Indeed, let $R_x \neq I_V$ and $R_y = I_V$ for some $x, y \in X$. By the transitivity of K there exists a permutation $k \in K$ such that $x^k = y$. Since $E^k = E$, we conclude that $R^k \in \mathcal{S}$. So all the basis matrices of W^X entering RR^k also belong to $\mathcal{S}$. On the other hand, since the cellular algebra W is primitive and nonregular, the product of two its basis matrices is a multiple of I_V iff both of them equal I_V. Let us apply this argument to matrices R_z and R_{z^k}, $z \in X$. Then since $R_x R_{x^k} = R_x R_y = R_x$ is not a multiple of I_V, there exists a matrix $S \in \mathcal{S}$ entering RR^k with $n_S < n_R$. However this contradicts the choice of R.

Since $R_x \neq I_V$, $x \in X$, and W is primitive and nonregular, each basis matrix of W enters $R_x^{i_x}$ for some positive integer i_x (see subsection 2.3). So any basis matrix of W^X enters R^i for some i (for instance we can take i to be the product of i_x over all $x \in X$). Since $R^i \circ E = R^i$, it means that $E = J_{V^X}$. So $W \uparrow K$ has only trivial equivalences and hence is primitive. ∎

By theorems 3.3 and 3.4 the non-Schurian primitive algebra $W = T(15)$ (see subsection 2.2) generates a series of non-Schurian primitive cellular algebras $W \uparrow K$ where K runs over all transitive groups. The smallest such an algebra not coinciding with W (on 225 points) arises for $K = Z_2$.

4 Representations, bases, groups

4.1 Let W be a cellular algebra on V. Since W is semisimple over $\mathbf{C}$, it is isomorphic to the direct product of full matrix algebras:

$$W = \prod_{P \in \mathrm{Spec}(W)} WP \cong \prod_{P \in \mathrm{Spec}(W)} \mathrm{Mat}_{[1,n_P]} \tag{6}$$

where $\mathrm{Spec}(W)$ is the set of all primitive central idempotents of W. It follows that

$$\dim_{\mathbf{C}}(W) = \sum_{P \in \mathrm{Spec}(W)} n_P^2. \tag{7}$$

Since $I_V = \sum_{P \in \mathrm{Spec}(W)} P$, we have the direct decomposition

$$\mathbf{C}^V = \sum_{P \in \mathrm{Spec}(W)} P\mathbf{C}^V. \tag{8}$$

where $\mathbf{C}^V$ is the linear space over $\mathbf{C}$ with base V. For each P the W-module $P\mathbf{C}^V$ is the direct sum of irreducible W-modules of dimension n_P over $\mathbf{C}$ isomorphic to each other. Let us denote their number by m_P. Then decomposition (8) implies that

$$n = \sum_{P \in \mathrm{Spec}(W)} m_P n_P. \tag{9}$$

The numbers m_p and n_P are called below the *multiplicity* of P and the *degree* of P respectively.

Let now W be a homogeneous cellular algebra. As it was shown in [6] the following inequality holds:

$$n_P \leq m_P, \qquad P \in \mathrm{Spec}(W). \tag{10}$$

Moreover, $n_P = m_P$ for all P iff W is regular.

Lemma 4.1 *Let $W \le \mathrm{Mat}_V$ be a homogeneous cellular algebra. Then*

$$\dim(WPv) = n_P^2$$

for all $P \in \mathrm{Spec}(W)$ and $v \in V$.

Proof. If $P \in \mathrm{Spec}(W)$, then $WP \cong \mathrm{Mat}_{[1,n_P]}$ and so the dimension of the algebra WP over $\mathbf{C}$ equals n_P^2. This implies the inequality

$$\dim(WPv) \le n_P^2, \qquad v \in V. \tag{11}$$

On the other hand, the set $X_v = \{Rv : \ R \in \mathcal{R}(W)\}$ is obviously a linearly independent subset of $\mathbf{C}^V$ for $v \in V$. Since W is homogeneous, we have $|X_v| = |\mathcal{R}(W)|$. So by (7)

$$\dim(Wv) = |X_v| = |\mathcal{R}(W)| = \dim(W) = \sum_{P \in \mathrm{Spec}(W)} n_P^2. \tag{12}$$

It follows from (8) that $\dim(Wv) = \sum_{P \in \mathrm{Spec}(W)} \dim(WPv)$. Thus (12) and (11) imply that $\dim(WPv) = n_P^2$ for all P.∎

We complete the subsection by remarking that given a homogeneous cellular algebra $W \le \mathrm{Mat}_V$, the matrix J_V/n is its primitive central idempotent of multiplicity and degree 1. It is called the *principal* idempotent of W.

4.2 The following notion is inspired by that of a base for permutation groups (see for instance [10]).

Definition 4.2 *A tuple $(v_1, \ldots, v_b) \in V^b$ is called a base (ordered) of a cellular algebra $W \le \mathrm{Mat}_V$ if $W_{v_1,\ldots,v_b} = \mathrm{Mat}_V$. We call this base irredundant if $\{v_i\} \notin \mathrm{Cel}(W_{v_1,\ldots,v_{i-1}})$ for all $i \in [1,b]$. Otherwise it is called redundant. A subset B of V is called a base of W if $W_{[B]} = \mathrm{Mat}_V$.*

In this section the term "base" always means ordered base. The minimum size of a base of W is denoted by $b(W)$. It is easy to see that $0 \le b(W) \le n-1$ for all W on V. The lower and upper bounds are attained exactly for the full matrix algebra Mat_V and the simplex $S(V)$ respectively. Given a permutation group $G \le \mathrm{Sym}(V)$ and a cellular algebra $W \le \mathrm{Mat}_V$, the following inequalities hold:

$$b(\mathrm{Aut}(W)) \le b(W), \quad b(G) \le b(\mathcal{Z}(G)) \tag{13}$$

where $b(G)$ is the minimum size of a base of G. The first follows from the equalities $\mathrm{Aut}(W)_{v_1,\ldots,v_b} = \mathrm{Aut}(W_{v_1,\ldots,v_b})$ and $\mathrm{Aut}(\mathrm{Mat}_V) = \{1\}$. The second is the consequence of the first and the obvious fact that $b(G) \le b(G')$ for $G \le G'$.

We do not develop the theory of cellular algebra bases in detail here but several remarks should be done. Using the notions of a base and a higher closure (see section 5) we can give a sufficient condition for a cellular algebra to be Schurian. Namely, it was proved in [4] that $(b+1)$-closure of a cellular algebra W is Schurian where $b = b(W)$. (In fact, the split number of W defined in that paper coincides with b.) We also mention the upper bound $b \le O(\sqrt{n}\log n)$ for a primitive cellular algebra W not coinciding with the simplex $S(V)$ proved in [1]. Finally, for cellular algebras arising from Hadamard matrices a logarithmic upper bound $b \le O(\log n)$ follows from [9].

4.3 It is easy to see (cf. (10) and below) that $b(W) \le 1 = m_P/n_P$ for a regular cellular algebra W and each $P \in \mathrm{Spec}(W)$. This observation can be generalized as follows.

Theorem 4.3 *Let $W \leq \mathrm{Mat}_V$ be a primitive cellular algebra. Then*

$$|B| \leq \frac{m_P}{n_P}$$

for each irredundant base B of W and each nonprincipal idempotent $P \in \mathrm{Spec}(W)$. In particular, $b(W) \leq \min_P(m_P/n_P)$ where P runs over the set of all such idempotents.

Remark 4.4 *The estimate of the theorem is sharp. The equality is attained, for instance, if W is a simplex or a regular algebra.*

Remark 4.5 *The primitivity of W is essential. Indeed, let $W = \mathcal{Z}(G)$ where G is the permutation group on 21 points defined by the action of the group PSL(3,2) on flags. Then W has an irredundant base of size 4 whereas $m_P/n_P = 3$ for its idempotent P of degree 2 and multiplicity 6.*

Proof. We start with two lemmas proved in [5]. To make the paper selfcontained we give the complete proofs of them.

Lemma 4.6 *Let W be a cellular algebra on V. For $A \in \mathrm{Mat}_V$ set*

$$\mathrm{Eq}(A) = \{(u,v) \in V \times V |\ Au = Av \neq 0\}.$$

Then $A \in W$ implies that the adjacency matrix of the relation $\mathrm{Eq}(A)$ belongs to W.

Proof. Let $\mathcal{R}$ be the standard basis of W. Since

$$\mathrm{Eq}\Big(\sum_{R\in\mathcal{R}} \alpha(R)R\Big) = \bigcup_{U\in\mathrm{Cel}(W)} \mathrm{Eq}\Big(\sum_{R\in\mathcal{R}_{V,U}} \alpha(R)R\Big) = \bigcup_{U\in\mathrm{Cel}(W)} \bigcap_{\alpha\neq 0} \mathrm{Eq}\Big(\sum_{R\in\mathcal{R}_{V,U},\alpha(R)=\alpha} R\Big),$$

it suffices to prove the lemma for a nonzero $A = \sum_R R$ where R runs over a subset of basis matrices of W contained in $\mathcal{R}_{V,U}$ for some $U \in \mathrm{Cel}(W)$. But for such an A we have

$$A^T A = d_{in}(A)\widetilde{A} + B,$$

where $d_{in}(A) = \sum_R d_{in}(R)$, $\widetilde{A}$ is the adjacency matrix of $\mathrm{Eq}(A)$ and $B = (B_{u,v})$ is a matrix with $0 \leq B_{u,v} < d_{in}(A)$ for all $u,v \in V$ and $B \circ \widetilde{A} = 0$. It follows that $\widetilde{A} \in W$.■

Lemma 4.7 *Let W be a cellular algebra on V and $A \in W$ be a matrix with pairwise distinct nonzero columns. Then $Av \in \sum_{i=1}^k Wv_i$ implies $\{v\} \in \mathrm{Cel}(W_{v_1,\ldots,v_k})$ where $v, v_1, \ldots, v_k \in V$.*

Proof. Let Av belong to $\sum_{i=1}^k Wv_i$ and $B \in \mathrm{Mat}_V$ be defined by

$$Bu = \begin{cases} Av, & \text{if } u \in \{v_1,\ldots,v_k\}; \\ Au, & \text{otherwise.} \end{cases}$$

Then it is easy to see that $B \in W_{v_1,\ldots,v_k}$. By lemma 4.6 we obtain that $\mathrm{Eq}(B)$ is an equivalence of $W_{v_1,\ldots,v_k}$. It follows from the hypothesis on A that $k \geq 1$ and the equivalence class modulo $\mathrm{Eq}(B)$ containing v_1 coincides with the set $\{v_1,\ldots,v_k,v\}$. Denote by U the cell of $W_{v_1,\ldots,v_k}$ containing v. Then $\{v_1\} \times U$ is a basis relation of the algebra $W_{v_1,\ldots,v_k}$. Thus $U \subset \{v_1,\ldots,v_k,v\}$, whence $U = \{v\}$.■

Let us turn into the theorem's proof. Let $B = \{v_1, \ldots, v_b)$ be an irredundant base of W and $P \in \mathrm{Spec}(W)$ be a nonprincipal idempotent. We will prove that the sum

$$L_k = \sum_{i=1}^{k} WPv_i \tag{14}$$

is direct for $k \in [0, b]$. It suffices to verify that $WPv_k \cap L_{k-1} = \{0\}$ for all $k \in [1, b]$. Let $Av_k \in L_{k-1}$ for some $A \in WP$. Then the choice of B guarantees by lemma 4.7 that the relation $\mathrm{Eq}(A)$ is different from the diagonal of $V \times V$. Hence either A has a zero column or $\mathrm{Eq}(A)$ is an equivalence of W by lemma 4.6. In both cases $A = 0$. In the first case this is the consequence of the homogeneity of W. In the second one the primitivity of W implies that $\mathrm{Eq}(A) = V \times V$. This means that A is a multiple of J_V and so $A = 0$ by the choice of P.

It follows from lemma 4.1 that $\dim(WPv_i) = n_P^2$ for all i. On the other hand, $L_b \subset P\mathbf{C}^V$ and $\dim(P\mathbf{C}^V) = m_P n_P$. Thus the direct decomposition (14) with $k = b$ gives the inequality

$$bn_P^2 \le m_P n_P,$$

which completes the proof.■

According to [12] the enveloping algebra $\mathrm{Env}(G)$ of a permutation group G coincides with the commutator algebra of $\mathcal{Z}(G)$ and vice versa. Moreover, $\mathrm{Spec}(\mathrm{Env}(G)) = \mathrm{Spec}(\mathcal{Z}(G))$. Besides, the degree n_π (multiplicity m_π) of an irreducible representation of G entering its permutation representation corresponding to an idempotent P coincides with m_P (respectively, n_P). Thus theorem 4.3 implies by (13) the following statement.

Corollary 4.8 *If G is a primitive permutation group, then*

$$b(G) \le \min_\pi (n_\pi / m_\pi)$$

where π runs over all nonprincipal irreducible representations of G entering its permutation representation.■

To formulate corollaries 4.9 and 4.12 let us define for a homogeneous cellular algebra $W \le \mathrm{Mat}_V$ the average degree d_{av} and the weighted average ratio r_{av} by

$$d_{av} = d_{av}(W) = \sum_{R \in \mathcal{R} \setminus \{I_V\}} d(R)/(|\mathcal{R}| - 1), \qquad r_{av} = r_{av}(W) = (\sum_P r_P n_P^2)/(\sum_P n_P^2) \tag{15}$$

where $\mathcal{R} = \mathcal{R}(W)$ is the standard basis of W, $r_P = m_P/n_P$ and P runs over all nonprincipal idempotents from $\mathrm{Spec}(W)$. Due to (3), (9) and (7) we have

$$d_{av}(t-1) = \sum_{R \ne I_V} d(R) = n - 1 = \sum_{P \ne J_V/n} m_P n_P = r_{av}(t-1)$$

where $t = |\mathcal{R}(W)| = \dim(W)$. Thus

$$d_{av}(W) = r_{av}(W) \tag{16}$$

Theorem 4.3 implies by (16) the following statement.

Corollary 4.9 *If W is a primitive cellular algebra, then $b(W) \le d_{av}(W)$.*■

4.4 There is an obvious upper bound

$$|\operatorname{Aut}(W)| \le n^{b(W)}$$

for the order of the automorphism group of a cellular algebra W. The following statement gives another estimate for it in primitive case.

Theorem 4.10 *Let W be a primitive cellular algebra on V. Then for each non-reflexive basis relation R of it there exists an irredundant base B of W such that*

$$|\operatorname{Aut}(W)| \le nd^{b-1}$$

where $d = d(R)$ is the degree of R and $b = |B|$ is the size of B.

Remark 4.11 *The statement of the theorem is still valid if W is an arbitrary cellular algebra, R is a connected relation on V being the union of basis relations S of W and $d = \max_S\{d_{in}(S), d_{out}(S)\}$.*

Proof. Let R be a nonreflexive basis relation of W. It follows from the primitivity of W that it is strongly connected. Therefore

$$\forall v_1, \ldots, v_i \in V: \quad W_{v_1,\ldots,v_i} \ne \operatorname{Mat}_V \;\Rightarrow\; \exists u \in X_i:\; R(u) \not\subset X_i$$

where $X_i = \{v \in V:\ \{v\} \in \operatorname{Cel}(W_{v_1,\ldots,v_i})\}$. This implies that there exists an irredundant base $B = (v_1, \ldots, v_b)$ such that $v_{i+1} \in V \setminus X_i$ and $R^T(v_{i+1}) \cap X_i \ne \emptyset$ for all $i \in [1, b-1]$. It follows that

$$[G_{v_1,\ldots,v_i} : G_{v_1,\ldots,v_{i+1}}] \le d, \qquad i \in [1, b-1]$$

where $G = \operatorname{Aut}(W)$. Since $[G : G_{v_1}] \le n$, the theorem follows.∎

Theorems 4.3 and 4.10 enable us due to (16) to estimate the order of the automorphism group of a primitive cellular algebra in terms of the ratios $r_P = m_P/n_P$ and the degrees $d(R)$.

Corollary 4.12 *Let $W \le \operatorname{Mat}_V$ be a primitive cellular algebra. Then*

$$|\operatorname{Aut}(W)| \le n(d_{min})^{d_{av}-1}, \quad |\operatorname{Aut}(W)| \le n(r_{av})^{r_{min}-1}$$

where d_{min} (resp. d_{av}) is the minimum (resp. average) degree of a nonreflexive basis relation of W and r_{min} (resp. r_{av}) is the minimum (resp. weighted average) ratio of a nonprincipal idempotent $P \in \operatorname{Spec}(W)$ (see (15)).∎

A famous Sims conjecture proved in [3] under the classification of finite simple groups (CFSG) states that $|G| \le nf(d)$ where $G \le \operatorname{Sym}(V)$ is a primitive permutation group and d is the minimum subdegree of G (coinciding with d_{min} of $\mathcal{Z}(G)$). The first inequality of corollary 4.12 with $W = \mathcal{Z}(G)$ provides a weaker upper bound for $|G|$. It would be interesting to estimate d_{av} by a function of d_{min}, which would give a new proof of the Sims conjecture without assuming the CFSG.

We complete the section by mentioning that according to theorem 4.2 of [6] the inequality $d_{min} \le 2^{O(m_P)}$ holds uniformly for all nonprincipal idempotent $P \in \operatorname{Spec}(W)$ of a primitive cellular algebra W. By theorems 4.10 and 4.3 this shows that

$$|\operatorname{Aut}(W)| \le n \cdot 2^{O(m_P^2/n_P)}$$

still uniformly for all such P.

5 Two-closed primitive cellular algebras

5.1 The notion of a m-closed cellular algebra was inroduced in [4] in connection with the Schurity problem. It goes back to [11] where a similar notion was defined in an algorithmic way. We start with the main definitions concerning higher closed cellular algebras.

Let W be a cellular algebra on V. For each positive integer m we set

$$\widehat{W} = \widehat{W}^{(m)} = [\underbrace{W \otimes \cdots \otimes W}_{m},\ \mathcal{Z}(\mathrm{Sym}(V), V^m)]$$

with $\mathrm{Sym}(V)$ acting on V^m coordinate-wise. We call the cellular algebra $\widehat{W}^{(m)} \le \mathrm{Mat}_{V^m}$ the *m-dimension extended algebra* of W. The group $\mathrm{Aut}(\widehat{W})$ acts faithfully on its invariant set $\Delta = \{(v, \ldots, v) :\ v \in V\}$. Moreover, the mapping $\xi : v \mapsto (v, \ldots, v)$ induces a permutation group isomorphism between $\mathrm{Aut}(W)$ and the constituent of $\mathrm{Aut}(\widehat{W})$ on Δ [1].

The important feature of the cellular algebra $\widehat{W}^{(m)}$ is the following possibility to extend the algebra W without changing its automorphism group. Set

$$W^{(m)} = ((\widehat{W}^{(m)})_\Delta)^{\xi^{-1}}.$$

We call $W^{(m)}$ the *m-closure* of W and say that W is *m-closed* if $W = W^{(m)}$. Each cellular algebra is certainly 1-closed. However it is not the case for $m \ge 2$ (see [4]). The following proposition describes the relationship between the m-closures and the Schurian closure $\mathrm{Sch}(W) = \mathcal{Z}(\mathrm{Aut}(W))$ of a cellular algebra W.

Proposition 5.1 ([4]) *For each cellular algebra W on V the following statements hold:*

(1) $\mathrm{Aut}(W^{(m)}) = \mathrm{Aut}(W)$ *for all* $m \ge 1$*;*

(2) $W = W^{(1)} \le \ldots \le W^{(n)} = \ldots = \mathrm{Sch}(W)$*;*

(3) $(W^{(m)})^{(l)} = W^{(m)}$ *for all* $l \in [1, m]$.■

Thus in a sense $W^{(m)}$ can be viewed as an approximation to $\mathrm{Sch}(W)$.

5.2 Below we restrict ourselves to the case $m = 2$. As we will see in the next subsection even in this case the properties of m-closed cellular algebras are different from those of general ones (at least in primitive case). We need a statement on the structure of 2-extended algebras which is a consequence of proposition 3.6 and lemmas 2.5 and 3.1 of [4].

Lemma 5.2 *Let W be a cellular algebra on V. Then*

(1) *A subset R of $V \times V$ is a basis relation of $W^{(2)}$ iff it is a cell of $\widehat{W}^{(2)}$.*

(2) *For each $v \in V$ the set $\widehat{W}_U = I_U \widehat{W} I_U$ where $U = U_v = \{v\} \times V$ and $\widehat{W} = \widehat{W}^{(2)}$ can naturally be viewed as a cellular algebra on the set U. Moreover, $(W_v)^{\zeta_U} \le \widehat{W}_U$ where the bijection $\zeta_U : V \to U$ is defined by $u^{\zeta_U} = (v, u)$.*

[1] It can be proved that in fact $\widehat{W} = [W \otimes \cdots \otimes W, I_\Delta]$.

(3) *Given two points v, v' belonging to the same cell of $W^{(2)}$ there exists a weak isomorphism $f : \widehat{W}_U \to \widehat{W}_{U'}$ with $U = U_v$ and $U' = U_{v'}$ such that $f(I_{(v,v)}) = I_{(v',v')}$ and $f(A^{\zeta_U}) = A^{\zeta_{U'}}$ for all $A \in W$.* ∎

If G is a permutation group on V, then obviously $\mathrm{Cel}(\mathcal{Z}(G_v)) = \mathrm{Orb}(G_v)$, $v \in V$, and $\mathcal{Z}(G_v) \cong \mathcal{Z}(G_{v'})$ for v, v' belonging to the same G-orbit (since $G_v \cong G_{v'}$ as permutation groups). The following statement generalizes these facts to 2-closed cellular algebras.

Lemma 5.3 *Let W be a 2-closed cellular algebra on V. Then*

(1) $\mathrm{Cel}(W_v) = \{R(v) : \ R \in \mathcal{R}\}$ *where $v \in V$ and $\mathcal{R}$ is the set of basis relations of W.*

(2) *Given two points v, v' belonging to the same cell of W there exists a weak isomorphism $\varphi = \varphi_{v,v'} : W_v \to W_{v'}$ such that $\varphi(I_v) = I_{v'}$ and $\varphi(A) = A$ for all $A \in W$.*

Proof. Let $v \in V$. Then $V = \cup_{R \in \mathcal{R}} R(v)$. So to prove (1) it suffices to check that $R(v)$ is a cell of W_v for all R. Let $R \in \mathcal{R}$. It follows from statement (1) of lemma 5.2 that $R \in \mathrm{Cel}(\widehat{W})$ where $\widehat{W} = \widehat{W}^{(2)}$. On the other hand, by the first part of statement (2) of lemma 5.2 we see that $\widehat{W}_U = I_U \widehat{W} I_U$ is a cellular algebra on the set $U = U_v$. Hence the set $U \cap R$ is a cell of $\widehat{W}_U$. So by the second part of statement (2) of lemma 5.2 there exists a cell $X \in \mathrm{Cel}(W_v)$ such that $X^\zeta \supset U \cap R$ where $\zeta = \zeta_U$. Since $(R(v))^\zeta = U \cap R$ we conclude that $X \supset R(v)$. However $R(v) \in \mathrm{Cel}^*(W_v)$. Thus $X = R(v)$ and so $R(v)$ is a cell of W.

To prove (2) set in the notation of lemma 5.2 $\widetilde{W} = (\widehat{W}_U)^{\zeta_U^{-1}}$ and $\widetilde{W}' = (\widehat{W}_{U'})^{\zeta_{U'}^{-1}}$. Then by statement (2) of that lemma $\widetilde{W}$ and $\widetilde{W}'$ are cellular algebras on V containing W_v and $W_{v'}$ respectively. Statement (3) of the same lemma implies then that the map

$$\varphi : \widetilde{W} \to \widetilde{W}', \quad \varphi(A) = (f(A^{\zeta_U})^{\zeta_{U'}^{-1}},$$

is a weak isomorphism from $\widetilde{W}$ to $\widetilde{W}'$. The algebra W_v (resp. $W_{v'}$) is the smallest cellular subalgebra of $\widetilde{W}$ (resp. $\widetilde{W}'$) containing W and I_v (resp. $I_{v'}$). Thus $\varphi(W_v) = W_{v'}$ by statement (3) of lemma 5.2. ∎

The lemma we proved has an interesting consequence concerning cellular algebras with nonreflexive basis relations of degree 1.

Corollary 5.4 *Let W be a 2-closed homogeneous cellular algebra on V and E be an equivalence of W being a union of basis relations of W of degree 1. Then $U \in \mathrm{Cel}(W[I_U])$ for each class $U \in V/E$.*

Proof. It follows from the hypothesis on E that $W_u = W_v = \widetilde{W}$ for all $u, v \in U$. Therefore the set $\{\varphi_{u,v} : \ u, v \in U\}$ is contained in the group of all weak isomorphisms from the algebra $\widetilde{W}$ to itself. So by statement (2) of lemma 5.3 this set is contained in the subgroup Φ preserving the standard basis of W. Hence the group Φ acts transitively on the set $\{I_u : \ u \in U\}$. This implies that U is a cell of the cellular algebra $\widetilde{W}^\Phi$ (see lemma 3.1). To complete the proof we note that $\widetilde{W}^\Phi \geq W[I_U]$, since the last algebra is the smallest cellular overalgebra of W containing I_U. ∎

5.3 In this subsection we present the properties of 2-closed primitive cellular algebras which generalize those of primitive permutation groups. In each case a "permutation group theorem" can be deduced from the corresponding "cellular algebra theorem" by using the simplest reasons such as: given a primitive permutation group G the cellular algebra $\mathcal{Z}(G)$ is also primitive, $\mathrm{Aut}(\mathcal{Z}(G))_v = G_v$ and so on. At the end of the subsection we give an example which shows that the hypothesis for a primitive algebra to be 2-closed is essential. We begin with the following characterization of 2-closed primitive algebras (cf. theorem 8.2 of [13]).

Theorem 5.5 *Let W be a 2-closed homogeneous cellular algebra on V. Then W is primitive iff the algebra W_v is a minimal cellular overalgebra of W for all $v \in V$.*

Proof. Let W be primitive and W' be a cellular algebra on V for which $W_v > W' \geq W$. Denote by U the cell of W' containing the point v. Then $U \neq \{v\}$ for otherwise $W' \geq W_v$. Thus there exists a nonreflexive basis relation R of W such that $R(v) \cap U \neq \emptyset$. On the other hand, by statement (1) of lemma 5.3 we conclude that $R(v) \in \mathrm{Cel}(W_v)$. This implies that $R(v)$ is contained in some cell of W'. By the choice of R this cell coincides with U, i.e. $R(v) \subset U$. Taking into account that the set $R \cap (U \times U)$ is the union of basis relations of the homogeneous component W'_U of the algebra W', we conclude that $R(u) \subset U$ for all $u \in U$. Since R is strongly connected (here we use the primitivity of W), it follows that $U = V$. Applying statement (1) of lemma 5.3 to $W_v = W'_v$ we see that each basis relation of W is also a basis relation of W'. Thus $W' = W$.

Conversely, let E be an equivalence of W, $E \neq V \times V$, and U be a class of E. Then we have the inclusion $W[I_U] \leq W_v$ for all $v \in U$. Since $W[I_U] \neq W$ (the choice of E), the minimality of W_v implies that $W[I_U] = W_v$ for all $v \in U$. By lemma 5.3 it follows that $d(R) = 1$ for all basis relation R of W such that $R \subset E$. Thus the equivalence E satisfies the hypothesis of corollary 5.4. So $U \in \mathrm{Cel}(W[I_U])$ which implies that $|U| = 1$. This shows that the equivalence E is trivial. ■

Another statement generalizes proposition 8.7 of [13].

Theorem 5.6 *Let W be a 2-closed primitive cellular algebra on V and u, v be different points of V. Then $W = W_u \cap W_v$ unless $W \cong \mathcal{Z}(Z_p)$ for a prime p.*

Proof. By theorem 5.5 we conclude that the cellular algebra $W_u \cap W_v$ coincides with either W or W_u. In the last case we also have $W_u = W_v$. Since W is 2-closed, it follows from lemma 5.3 that the basis relation R of W for which $v \in R(u)$ is of degree 1. So $W \cong \mathcal{Z}(Z_p)$ by lemma 2.1. ■

The following theorem generalizes theorem 17.6 of [13] on the faithful constituents of a one-point stabilizer of a primitive permutation group.

Theorem 5.7 *Let W be a 2-closed primitive cellular algebra on V. Let $v \in V$ and $B \in \mathrm{Cel}^*(W_v)$, $B \neq \{v\}$ be a nonempty subset of V containing the union of all cells U of W_v such that the homogeneous component of W_v corresponding to U is imprimitive. Then B is a base of W.*

Proof. Denote by $\mathcal{R}$ and $\mathcal{R}_v$ the sets of basis relations of W and W_v respectively.

Lemma 5.8 *Let W be a 2-closed primitive nonregular cellular algebra on V and $v \in V$. Then there exists a nonreflexive basis relation R of W such that*

$$\forall S \in \mathcal{R}: \quad S(v) \times R(v) \in \mathcal{R}_v \ \Leftrightarrow \ S = \Delta$$

where $\Delta = \Delta_V = \{(v, v) : \ v \in V\}$ is the diagonal of $V \times V$.

Proof. Choose $R \in \mathcal{R}$ to be a nonreflexive basis relation of maximal degree. Suppose that $S(v) \times R(v) \in \mathcal{R}_v$ for some $S \in \mathcal{R}$. By lemma 5.3 the sets $S(v)$ and $R(v)$ are cells of W_v. So there exists $T \in \mathcal{R}$ such that $T \supset S(v) \times R(v)$. Clearly, $T(u) \supset R(v)$ for all $u \in S(v)$. So

$$T(u) = R(v), \quad u \in S(v) \tag{17}$$

by the choice of R. The equivalence on V defined by $u \sim w$ iff $T(u) = T(w)$ coincides with Δ by the primitivity of W and lemma 4.6 (applied to the adjacency matrix of the relation T). So $d(S) = |S(v)| = 1$ due to (17)). Thus $S = \Delta$, for otherwise W is regular by lemma 2.1. Since the converse implication follows from lemma 5.3, we are done.■

Let us turn into the theorem's proof. Without loss of generality we assume that W is not regular and check first that B is a base of W_v. For two cells U, U' of W_v we define the equivalence $E(U, U')$ on U by

$$E(U,U') = \{(u_1, u_2) \in U \times U :\ S(u_1) = S(u_2) \text{ for all } S \in \mathcal{R}_v,\ S \subset U \times U'\}.$$

Then it is easy to see that

$$E(U,U') = \Delta_U \ \Rightarrow\ W_v(U,U') = \mathrm{Mat}_U \tag{18}$$

where $W_v(U,U')$ is the restriction of $(W_v)_{[U']}$ to U. Choose R as in lemma 5.8 and set $X = R(v)$. By lemma 5.3 the set X is a cell of W_v. Suppose first that $X \subset B$. Then it follows from theorem's hypothesis that $E(U,X)$ is either $U \times U$ or Δ_U for all cell $U \not\subset B$. But if $U \neq \{v\}$ the first case is impossible by lemma 5.8 and so the statement follows from (18). Suppose now that $X \not\subset B$ and hence the homogeneous component of W_v corresponding to X is primitive. Then the above argument shows that $E(X,U') = \Delta_X$ for all cell $U' \subset B$. So by (18) we can assume that $X \subset B$ which returns us to the previous case.

To complete the proof we note that B is in fact a base of W. Otherwise, the cell U of $W_{[B]}$ containing v is not a singleton. Since $R(u) = R(v)$ for all $u \in U$ and any basis relation R of W with $R(v) \subset B$, the above argument shows that W has a nontrivial equivalence, which contradicts the primitivity of W.■

To see that the hypothesis for W to be 2-closed is essential let us consider the non-Schurian primitive cellular algebra $W = T(15)$ on $V = [1, 15]$ defined in subsection 2.2. A straightforward computation shows then that there exists a point $v_0 \in V$ such that

(1) W_{v_0} has three cells: two cells of size 7 and one cell of size 1;

(2) if $v \neq v_0$, then W_v has seven cells: four cells of size 3 and three cells of size 1;

(3) $W_v > W_{v_0}$, if $v \neq v_0$.

Choose an arbitrary point $v \in V$ different from v_0. Then it is easy to see that statements (1) and (2) of lemma 5.3 do not hold for the point v and the pair (v_0, v) respectively. So W can not be 2-closed. (In fact, $W^{(2)} = \mathrm{Sch}(W)$). On the other hand, the conclusion of theorem 5.5 is not true since W_v is not a minimal overalgebra of W by property (3). It can be shown in a similar way that for the algebra W the conclusions of theorems 5.6 and 5.7 are also false.

5.4 It is well-known that each 3/2-transitive permutation group is either primitive or a Frobenius group (see theorem 10.4 in [13]). The following statement is a generalization of this result to cellular algebras (see also [11]).

Theorem 5.9 *Let W be a 2-closed homogeneous cellular algebra on V. Suppose that $d(R) = d(S)$ for all nonreflexive basis relations R, S of W. Then either W is primitive or each irredundant base of W is of size at most 2.*

Remark 5.10 *It follows from theorem 1.3 of [4] that in the second case the algebra W is Schurian provided that it is 3-closed.*

Proof. Let W be an imprimitive cellular algebra with $\mathcal{R}$ as the set of basis relations, E be an equivalence of W and $U \in V/E$. Then

$$|U| = \sum_{R \in \mathcal{R}, R \subset E} d(R) = d(a-1) + 1$$

where d is the degree of any nonreflexive basis relation of W and a is the cardinality of the set under the sum. So

$$\mathrm{GCD}(d, |U|) = 1. \tag{19}$$

For $v \in V$ set $\mathcal{S} = \mathcal{S}(v, U) = \{S \in \mathcal{R} : \ S(v) \cap U \neq \emptyset\}$. Then the sets $\mathcal{S}(v, U)$ for different U either coincide or disjoint (see [11]). So the set $\cup_{S \in \mathcal{S}} S(v)$ is the union of classes of E and hence

$$d|\mathcal{S}| = \sum_{S \in \mathcal{S}} d(S) = |U| l$$

for some positive integer l. It follows from (19) that $|U|$ divides $\mathcal{S}$. Since $|\mathcal{S}| \leq |U|$, we conclude that $|\mathcal{S}| = |U|$. Thus

$$|R(v) \cap U| \leq 1, \qquad U \in V/E, \ v \in V \setminus U \tag{20}$$

for all $R \in \mathcal{R}$.

Let us show that $W_{u,v} = \mathrm{Mat}_V$ for distinct $u, v \in V$. Denote by U the class of E containing u. Then it suffices to prove that $d_{in}(R) = d_{out}(R) = 1$ for all $R \in \mathcal{R}_u$ contained in $(V \setminus \{u\}) \times (V \setminus \{u\})$ where $\mathcal{R}_u$ is the set of basis relations of W_u. Since $W_u \geq W[I_U]$, we see that for each cell $X \in \mathrm{Cel}(W_u)$ either $X \subset U$ or $X \cap U = \emptyset$. Let X and X' be cells of W_u such that $X \subset U \setminus \{u\}$ and $X' \cap U = \emptyset$. Then by (20) for an arbitrary $v \in X'$ the pairs (v, w) and (v, w') with distinct $w, w' \in X$ belong to distinct basis relations of W. Therefore the number of basis relations of W_u contained in $X' \times X$ equals $|X|$. On the other hand, since W is 2-closed, we have by lemma 5.3 that $|X| = |X'| = d$. So we conclude that

$$\forall R \in \mathcal{R}_u : \ R \subset (U \setminus \{u\}) \times (V \setminus U) \ \Rightarrow \ d_{in}(R) = d_{out}(R) = 1.$$

To complete the proof we note that each basis matrix of W_u with support in $(U \setminus \{u\}) \times (U \setminus \{u\})$ or $(V \setminus U) \times (V \setminus U)$ can be written as the product of two basis matrices of W_u with support in $(U \setminus \{u\}) \times (V \setminus U)$ and $(V \setminus U) \times (U \setminus \{u\})$.■

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