NP-Hardness of the Bandwidth Problem on Dense Graphs

(Revised)

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Abstract

The *bandwidth problem* is the problem of numbering the vertices of a given graph G such that the maximum difference between the numbers of adjacent vertices is *minimal*. The problem has a long and varied history and is known to be NP-hard Papadimitriou [Pa 76]. Recently for δ-dense graphs a constant ratio approximation algorithm for this problem has been constructed in Karpinski, Wirtgen and Zelikovsky [KWZ 97]. In this paper we prove that the bandwidth problem on the dense instances remains NP-hard.

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1 Introduction

The bandwidth problem on graphs has a very long and interesting history cf. [CCDG 82].

Formally the bandwidth minimization problem is defined as follows. Let $G = (V, E)$ be a simple graph on n vertices. A numbering (or layout) of G is a one-to-one mapping $f : V \to \{1, ..., n\}$. The bandwidth $B(f, G)$ of this numbering is defined by

$$B(f, G) = \max\{|f(v) - f(w)| : \{v, w\} \in E\},$$

the greatest distance between adjacent vertices in G corresponding to f. The bandwidth $B(G)$ is then

$$B(G) = \min_{f \text{is a numbering of } G} \{B(f, G)\}$$

Clearly the bandwidth of G is the greatest bandwidth of its components.

The problem of finding the bandwidth of a graph is NP-hard [Pa 76], even for trees with maximum degree 3 [GGJK 78]. The general problem is not known to have any sublinear n^{ϵ}-approximation algorithms. There are only few cases where we can find the optimal layout in polynomial time. Saxe [Sa 80] designed an algorithm which decides whether a given graph has bandwidth at most k in time $O(n^k)$ by dynamic programming. Bandwidth two can be checked in linear time [GGJK 78]. Kratsch [Kr 87] introduced an exact $O(n^2 \log n)$ algorithm for the bandwidth problem in interval graphs. Smithline [Sm 95] proved that the bandwidth of the complete k-ary tree $T_{k,d}$ with d levels and k^d leaves is exactly $\lceil k(k^d - 1)/(k - 1)(2d) \rceil$. Her proof is constructive and contains a polynomial time algorithm, for this task. For caterpillars [HMM 91] found a polynomial time $\log n$-approximation algorithm. A caterpillar is a special kind of a tree consisting of a simple chain, the body, with an arbitrary number of simple chains, the hairs, attached to the body by coalescing an endpoint of the added chain with a vertex of the body. Karpinski, Wirtgen and Zelikovsky [KWZ 97] designed a 3-approximation algorithm for δ-dense graphs.

Definition 1 ([AKK 95]) *We call a graph dense, if the number of edges is in $\Omega(n^2)$. A graph G is δ-dense, if the minimum degree $\delta(G)$ is at least δn. We call it everywhere dense, if it is δ-dense for some $\delta > 0$ in $O(1)$.*

In this paper we show that the bandwidth problem on dense graphs is NP-hard, answering the question raised in [KWZ 97].

This paper is organized as follows. In Section 2 we introduce some notations in graph theory related to the bandwidth and discuss some known results of [ACP 87] [BGHK 95] [KKM 96]. In section 3 we relate the results of section 2 to the bandwidth problem in everywhere dense graphs and prove its NP-hardness.

2 Related Notations and Known Results

The class of *k-trees* is defined recursively as follows:

1. The complete graph on k vertices is a k-tree.
2. Let G be a k-tree on n vertices, then the graph constructed as follows is also a k-tree: add a new vertex and connect it to all vertices of a k-clique of G, and only to these vertices.

Any subgraph of a k-tree is called *partial k-tree*. Arnborg et al. showed in [ACP 87] that PARTIAL-k-TREE is NP-complete. PARTIAL-k-TREE is the problem given a graph G and an integer k, decide whether G is a partial k-tree or not.

A *tree decomposition* of a graph $G = (V, E)$ is a pair $(\{X_i | i \in I\}, T = (I, F))$, where T is a tree and $\{X_i\}$ is a set of subsets of V, such that

1. $\bigcup_{i \in I} X_i = V$
2. For all $\{u, v\} \in E$, there is an $i \in I$ with $u, v \in X_i$
3. For all $i, j, k \in I$, if j is on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$.

The *treewidth* $tw((\{X_i\}, T), G)$ of a tree decomposition $(\{X_i\}, T)$ is defined by

$$tw((\{X_i\}, T), G) = \max_i |X_i| - 1$$

The *treewidth* $tw(G)$ of a graph G is then

$$tw(G) = \min_{(\{X_i\}, T)} tw((\{X_i\}, T), G)$$

Between the treewidth of a graph and the smallest k such that G is a partial k-tree exists the following well known connection:

Lemma 2 *For $k \geq 1$ the treewidth of a graph G is at most k if and only if G is a partial k-tree. Thus $tw(G)$ equals to the smallest k such that G is a partial k-tree.*

PROOF: See, for example, [Le 90]. ■

There is also a connection between the bandwidth and the treewidth of cobipartite graphs as showed in [KKM 96]. We call a graph *cobipartite* if it is the complement of a bipartite graph.

Lemma 3 ([KKM 96]) *Let G be a cobipartite graph. Then*

$$B(G) = tw(G)$$

Using Lemma 2 we get

Corollary 4 *Let G be a cobipartite graph. Then $B(G)$ equals to the smallest k such that G is a partial k-tree.*

In section 3 we will have a closer look to the proof of NP-hardness of PARTIAL-k-TREE and prove that the instance for PARTIAL-k-TREE constructed there, is everywhere dense and cobipartite. Thus it is easy to show that the bandwidth problem on everywhere dense graphs is NP-hard.

3 NP-Hardness for Everywhere Dense Graphs

First of all we sketch the proof of NP-hardness of PARTIAL-k-TREE proposed in [ACP 87] to show that the constructed instance is a everywhere dense cobipartite graph. By the results stated in section 2 the NP-hardness of bandwidth in everywhere dense graphs follows.

Theorem 5 ([ACP 87]) PARTIAL-k-TREE *is NP-hard.*

PROOF: (Sketch) Let $G = (V, E)$ be a input graph of the NP-complete MINIMUM CUT LINEAR ARRANGEMENT (MCLA) problem (for the proof of NP-completeness see [GJ 79] [GT44]): given G and a positive integer k, does there exist a numbering f of V, such that

$$c(f, G) = \max_{1 \le j < n} |\{\{u, v\} \in E | f(u) \le j < f(v)\}| \le k$$

We will construct a bipartite graph $G' = (A \cup B, E')$. The vertices are defined as follows:

- Each $v \in V$ is represented by $\Delta(G) + 1$ vertices in A, building the set A_v (We denote by $\Delta(G)$ the maximum vertex degree in G) and $\Delta(G) - \deg(v) + 1$ vertices in B, building the set B_v.
- For each edge $e \in E$ there are two vertices in B. They are denoted by B_e.

There are two different edge types in E':

- All vertices in A_v are connected to both vertices in B_e, if $v \in e$.
- All vertices of A_v are connected with all vertices in B_v.

Now define G'' to be G' after inserting all edges in A and B. Arnborg et al. showed the following connection: *G has a minimum linear cut value k with respect to some numbering f, if and only if the corresponding graph G'' is a partial k'-tree for*

$k' = (\Delta(G)+1)(|V|+1)+k-1$. Since the construction of G'' is polynomial, it follows that PARTIAL-k'-TREE is NP-hard. ■

As a corollary we get the following theorem.

Theorem 6

The bandwidth problem on everywhere dense graphs is NP-hard.

PROOF: Observe that the instance for PARTIAL-k-TREE constructed in the proof of Theorem 5 is cobipartite. Further it is at least 1/2-dense, since the sets A and B build cliques and $|A| = |B|$:

$$\begin{aligned} |A| &= (\Delta(G)+1)|V| \\ (n=|V|) &= \Delta(G)n + n \\ &= \Delta(G)n + n - \sum_{v\in V} \deg(v) + 2|E| \\ &= \sum_{v\in V} (\Delta(G) - \deg(v) + 1) + 2|E| \\ &= |B| \end{aligned}$$

Applying Corollary 4 it follows, since G is cobipartite that the bandwidth on everywhere dense graphs is NP-hard. ■

4 Open Problems

An important computational problem remains open about the approximation hardness of the bandwidth on the dense instances.

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